Rabinowitz Floer Homology for Tentacular Hamiltonians

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CHAPTER

THREE

TENTACULAR HAMILTONIANS

Nad zrębem planety,
Pośród gwiezdnej nocy,
Szereg alefów w nieskończoność pełznie.
I nieskończoność unieskończoniona
Zamiera w sobie, przez siebie zdradzona.
Kłęby Tytanów i rogate widma
Sypią gwiazd roje
W wydarte otchłanie.
Myśl w własne wątpia zapuściła szpony
I gryzie siebie w swej własnej otchłani. [1]

Stanisław Ignacy Witkiewicz,
Hop, szklankę piwa!

In this chapter we will introduce a class of Hamiltonians on the symplectic manifold $(\mathbb{R}^{2n}, \omega_0)$, which include examples of non-compact energy hypersurfaces and prove that for this class the Rabinowitz Floer homology as stated in previous chapter is well defined, independent on the choices and invariant under compact perturbations. In particular, we will show that the class is open under compact perturbations, that all the Hamiltonians from the class satisfy conditions (SG), (PO) and (BD), whereas condition (MB) is satisfied generically. We will also show that for a small enough perturbation of the Hamiltonian and the almost complex structure, the assumptions of Theorem 4 are satisfied, which implies that for this class of Hamiltonians the definition of Rabinowitz Floer homology is not only well defined, but also independent of choices and invariant under compactly supported homotopies.

[1] "At the planet's edge, amidst a starry night, a series of alephs crawls to infinity. And the infinitized infinity is petrified by its own betrayal. Tangles of Titans and horned spectres scatter bevies of stars into the abyss. The thought dug its claws into its entrails and gnaws at itself in its own abyss." *Heigh-ho, a glass of beer!* by Stanisław Ignacy Witkiewicz, translated by J.J. Wiśniewska.

Let us define the special class of Hamiltonians we will be working with

Definition 3.1. ***(Tentacular Hamiltonians)***
For a smooth Hamiltonian $H : \mathbb{R}^{2n} \to \mathbb{R}$ we can state the following properties:

H1 *There exists a global Liouville vector field $X^\dagger$ and constants $c_1, c_2 > 0$, $c_3 \geq 0$ such that for all $x \in \mathbb{R}^{2n}$ the following holds true*

$$\begin{aligned} \|X^\dagger_x\| &\leq c_1(\|x\| + 1), \\ dH_x(X^\dagger) &\geq c_2\|x\|^2 - c_3. \end{aligned}$$

H2 *H grows at most quadratically at infinity*

$$\sup_{x\in\mathbb{R}^{2n}} \|D^3 H_x\| \cdot \|x\| < L.$$

H3 *There exist constants $c_4, c_5, \nu > 0$ and a Liouville vector field $X^\ddagger$ defined on $H^{-1}((-\nu, \nu))$, such that*

$$\begin{aligned} \|X^\ddagger(x)\| &\leq c_4(\|x\|^2 + 1) \qquad \forall\, x \in \mathbb{R}^{2n}, \\ c_5 := \inf_{H^{-1}((-\nu,\nu))} & dH(X^\ddagger) > 0. \end{aligned}$$

H4 *There exist a coercive function $\mathcal{F} : \mathbb{R}^{2n} \to \mathbb{R}$ such that the set*

$$\{x \in \mathbb{R}^{2n} \mid d_x(d\mathcal{F}(X^H))(X^H(x)) \leq 0\} \cap \{x \in \mathbb{R}^{2n} \mid d_x\mathcal{F}(X^H) = 0\} \cap H^{-1}(0)$$

is compact in $\mathbb{R}^{2n}$.

We will denote by $\mathscr{H}$ the set of all smooth Hamiltonians satisfying Property (SG) and all the above Properties and call them ***tentacular*** *Hamiltonians.*

Let us analyze the consequences of the above properties. First observe that Property (H1) of the definition implies linear growth of the gradient of H

$$\begin{aligned} \|\nabla H(x)\| &\geq \frac{c_2}{c_1}(\|x\| - h_1') \qquad \forall\, x \in \mathbb{R}^{2n} \\ \text{where} \qquad h_1' &= 1 + \frac{|c_2 - c_3|}{c_2}. \end{aligned}$$

In a similar way Property (H2) induces quadratic behavior of the Hamiltonian for all $x \in \mathbb{R}^{2n}$ we have

$$\|Hess_x H\| \leq M, \tag{3.1}$$

$$\|\nabla H(x)\| \leq h_1 + M\|x\|, \tag{3.2}$$

$$|H(x)| \leq h_0 + h_1\|x\| + \frac{1}{2}M\|x\|^2, \tag{3.3}$$

$$\text{where} \qquad M := \|Hess_0 H\| + L, \qquad h_0 := |H(0)|, \qquad h_1 := \|\nabla H(0)\|.$$

Moreover, note that Property (H3) ensures that 0 is a regular value of H and $H^{-1}(0)$ is of contact type. Finally, by the argumentation from the proof of Corollary 2.28 we see that Property (H4) implies that all the non-degenerate periodic orbits of the Hamiltonian vector field on $H^{-1}(0)$ are contained in a compact subset of $H^{-1}(0)$. In other words Property (H4) implies Property (PO+). Moreover, if we apply a compact perturbation, then the perturbed Hamiltonian will have all the periodic orbits also bounded.

This way we can see that every tentacular Hamiltonian satisfies three of the five properties needed to construct Rabinowitz Floer homology as stated in Theorem 1, namely properties (SG), (CT) and (PO). The last two properties are more challenging - to prove that property (MB) is satisfied for generic compact perturbations of tentacular Hamiltonians, one has first has to prove openness of the set of tentacular Hamiltonians under compact perturbations as it is done in Lemma 3.2 and then apply Corollary 2.28. Finally, the proof of property (BD) is done in Theorem 7 and occupies most of this chapter.

Observe that from the presented Properties (SG) is a strictly topological property, (H2) is strictly analytic, whereas Properties (H1), (H3) and (H4) take into the account the symplectic structure of the manifold.

One will find more discussion on the Properties and examples of tentacular Hamiltonians in Chapter 4. To get a feeling about tentacular Hamiltonians one can keep in mind our "toy example" of a tentacular Hamiltonian, the hyperboloid in $\mathbb{R}^4$

$$H(q_1, q_2, p_1, p_2) := \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 - q_2^2 - 1).$$

3.1 Rabinowitz Floer Homology for tentacular Hamiltonians

The following Theorem is the main result of this thesis. It proves that the Rabinowitz Floer homology is well defined on the whole set of tentacular Hamiltonians by incorporating the results for the tentacular Hamiltonians into the framework of Chapter 2. The proof refers to results, which are proven further on in this chapter, the most important being Theorem 7 and lemmas in Section 3.9.

Theorem 6. *For all tentacular Hamiltonians the Rabinowitz Floer homology is well defined. Moreover, if we have a smooth family $\{H_\chi\}_{\chi\in[0,1]} \in \mathscr{H}$, such that*

$$H_\chi - H_0 \in C_c^\infty(\mathbb{R}^{2n}),$$

then $RFH(H_0)$ and $RFH(H_1)$ are isomorphic

$$RFH(H_0) \cong RFH(H_1).$$

Note that we use here notation $RFH(H)$ rather than $RFH(H, J\mathbb{R}^{2n})$ introduced in Theorem 1 omitting the symplectic manifold $(\mathbb{R}^{2n}, \omega_0)$ from the notation, since we defined tentacular Hamiltonians only on this space, and omitting the almost complex structure form the notation, since in the proof below we show that Rabinowitz Floer homology for tentacular Hamiltonians is independent of the choice of J.

Outline of the proof: The proof consists of two parts. First we will show that there exists a dense subset $\mathscr{H}^{reg} \subseteq \mathscr{H}$ for which conditions (SG), (CT), (MB), (PO) and (BD) are satisfied and that Rabinowitz Floer homology is well defined for all $H \in \mathscr{H}^{reg}$. Later we will show that the Rabinowitz Floer homology can be extended to $\mathscr{H}$ in such a way that it is invariant under compact perturbations within $\mathscr{H}$. As a result the definition does not depend on the choice of the regular quintuple and can be extended on the whole set of $\mathscr{H}$.

By Lemma 3.2 the set $\mathscr{H}$ is open under compact perturbations. In other words, for every $H \in \mathscr{H}$ the set

$$\{h \in C_c^\infty(\mathbb{R}^{2n}) \mid H + h \in \mathscr{H}\}$$

is open in $C_c^\infty(\mathbb{R}^{2n})$.

By Corollary 2.28 Property (H4) implies condition (PO+). In fact Property (H4) assures that condition (PO+) persists under compact perturbations. Corollary 2.28 shows us how from the fact that condition (PO+) persists under compact perturbations one can deduce that the set

$$\mathscr{H}^{reg} := \{H \in \mathscr{H} \mid H \text{ satisfies Property MB}\}$$

is dense in $\mathscr{H}$. In other words, for every Hamiltonian $H \in \mathscr{H}$ the set

$$\{h \in C_c^\infty(\mathbb{R}^{2n}) \mid H + h \in \mathscr{H} \;\&\; H + h \text{ satisfies Property MB}\}$$

is dense in

$$\{h \in C_c^\infty(\mathbb{R}^{2n}) \mid H + h \in \mathscr{H}\}$$

in the $C_c^\infty(\mathbb{R}^{2n})$ topology.

On the other hand, from property (PO+), using Proposition 2.24, one can deduce that for every $H \in \mathscr{H}$ there exists an open, precompact subset $\mathcal{V} \subseteq M$ and a constant $\mathfrak{y} > 0$, such that for all $(v, \eta) \in Crit(\mathcal{A}^H) \setminus (\mathcal{A}^H)^{-1}(0)$ one has:

$$v(S^1) \times \{\eta\} \subseteq \mathcal{V} \times \big((-\infty, \mathfrak{y}) \cup (\mathfrak{y}, +\infty)\big).$$

In other words for every $H \in \mathscr{H}$, all non-degenerate critical points of $\mathcal{A}^H$ are contained in the set $\mathcal{V} \times \big((-\infty, \mathfrak{y}) \cup (\mathfrak{y}, +\infty)\big)$, hence for every pair of connected components of $\Lambda^-, \Lambda^+ \subseteq Crit(\mathcal{A}^H)$ and every

$$J \in \mathscr{J}\Big(\mathbb{R}^{2n}, \omega_0, \mathcal{V} \times \big((-\infty, \mathfrak{y}) \cup (\mathfrak{y}, +\infty)\big)\Big)$$

every Floer trajectory in the corresponding moduli space $u \in \mathscr{M}(\Lambda^-, \Lambda^+)$ passes through $\mathcal{V} \times \big((-\infty, \mathfrak{y}) \cup (\mathfrak{y}, +\infty)\big)$, that is:

$$u(\mathbb{R} \times S^1) \cap \Big(\mathcal{V} \times \big((-\infty, \mathfrak{y}) \cup (\mathfrak{y}, +\infty)\big)\Big) \neq \emptyset.$$

This assures that the last part of Property (BD) is satisfied.

Having showed that all the Hamiltonians $H \in \mathscr{H}$, satisfy conditions (PO), (SG), (H1), (H2) and (H3) one can apply Theorem 7 to the "constant" homotopy $\Gamma = Id_{(H,J)} = \{H, J\}_{s\in\mathbb{R}}$ where J is any family of almost complex structures from the set

$$J \in \mathscr{J}\Big(\mathbb{R}^{2n}, \omega_0, \mathcal{V} \times \big((-\infty, \mathfrak{y}) \cup (\mathfrak{y}, +\infty)\big)\Big).$$

Then the condition (3.6) is trivially satisfied and therefore all the assumptions of Theorem 7 are satisfied and we can conclude that Property (BD) is satisfied for every $H \in \mathscr{H}$.

Having established that Hamiltonians in the set $\mathscr{H}^{reg}$ satisfy the conditions (SG), (CT), (PO), (MB) and (BD) we can apply Theorem 2 to conclude that for every $H \in \mathscr{H}^{reg}$ the set of regular almost complex structures

$$\mathscr{J}^{reg}(H) \subseteq \mathscr{J}\Big(\mathbb{R}^{2n}, \omega_0, \mathcal{V} \times \big((-\infty, \mathfrak{y}) \cup (\mathfrak{y}, +\infty)\big)\Big)$$

is generic. Therefore, for $H \in \mathscr{H}^{reg}$ one can choose a regular quintuple (H, J, N, f, g) with $J \in \mathscr{J}^{reg}(H)$ and by Theorem 1 one can define the corresponding Rabinowitz Floer homology. In fact, by Proposition 2.13 and Lemma 3.21 for every $H \in \mathscr{H}^{reg}$ the Rabinowitz Floer homology is independent of the choices of N, f, g and J and therefore will be denoted by

$$RFH(H).$$

Now we would like to extend the definition of Rabinowitz Floer homology to the entire set of tentacular Hamiltonians. By Lemma 3.22 for every $H \in \mathscr{H}$ and every compact subset $K \subseteq \mathbb{R}^{2n}$ there exists a neighborhood $\widetilde{\mathscr{O}}(H)$ of 0 in $C_0^\infty(K)$, such that for every pair $h_1, h_2 \in \mathscr{O}(H)$, such that $H + h_1, H + h_2 \in \mathscr{H}^{reg}$ the corresponding Rabinowitz Floer homologies are isomorphic, i.e.

$$RFH(H + h_1) \cong RFH(H + h_2).$$

Therefore we can define

$$RFH(H) := RFH(H + h) \qquad h \in \widetilde{\mathscr{O}}(H) \tag{3.4}$$

and the definition does not depend on the choice of $h \in \widetilde{\mathscr{O}}(H)$, nor on the choice of the compact subset K.

Finally, we will show that the Rabinowitz Floer homology is invariant under compactly supported homotopies. Let $\{H_\chi\}_{\chi\in[0,1]} \in \mathscr{H}$ be a smooth family, such that

$$H_\chi - H_0 \in C_c^\infty(\mathbb{R}^{2n}) \qquad \forall \chi \in [0, 1].$$

By Theorem 2.5 in [22] there exists a compact subset $K \subseteq \mathbb{R}^{2n}$, such that

$$H_\chi - H_0 \in C_0^\infty(K) \qquad \forall \chi \in [0, 1].$$

For every $\chi \in [0,1]$ let $\widetilde{\mathscr{O}}(H_\chi) \subseteq C_0^\infty(K)$ be the open set corresponding to H_χ as provided by Lemma 3.22. Then the family

$$\{H_\chi - H_0 + \widetilde{\mathscr{O}}(H_\chi)\}_{\chi \in [0,1]}$$

is an open cover in $C_0^\infty(K)$ of the compact set

$$\bigcup_{\chi \in [0,1]} (H_\chi - H_0)$$

and as such it admits a finite subcover

$$\{H_{\chi_k} - H_0 + \widetilde{\mathscr{O}}(H_{\chi_k})\}_{k=1}^K.$$

Without loss of generality we can assume that

$$\begin{aligned} & H_0 - H_{\chi_1} \in \widetilde{\mathscr{O}}(H_{\chi_1}), \\ k = 1, \dots K-1, \qquad & (H_{\chi_k} + \widetilde{\mathscr{O}}(H_{\chi_k})) \cap (H_{\chi_{k+1}} + \widetilde{\mathscr{O}}(H_{\chi_{k+1}})) \neq \emptyset, \\ & H_1 - H_{\chi_K} \in \widetilde{\mathscr{O}}(H_{\chi_K}). \end{aligned}$$

Then by (3.4) we have

$$RFH(H_0) = RFH(H_{\chi_1}) = \dots = RFH(H_{\chi_K}) = RFH(H_1).$$

□

Theorem 6 proves that the Rabinowitz Floer homology for tentacular Hamiltonians is well defined and invariant under compactly supported homotopies. In particular, for every $H \in \mathscr{H}$ it is constant on every connected component of

$$(H + C_c^\infty(\mathbb{R}^{2n})) \cap \mathscr{H}.$$

Let us take a closer look at the definition of tentacular Hamiltonians, analyze what role each property plays in the construction of Rabinowitz Floer homology and discuss the ideas how the theory could be extended to a larger class of Hamiltonians. First observe that the main theorem which assures the boundedness of Floer trajectories, namely Theorem 7 does not require property (H4) nor (PO+), but only property (PO) instead. Therefore Theorem 7 can be applied to a broader class of Hamiltonians than only those defined in Definition 3.1, namely those which satisfy properties (H1), (H2), (H3) and (PO), but not necessarily (H4). However, boundedness of Floer trajectories, i.e. condition (BD) is not sufficient to construct Rabinowitz Floer homology - one needs also to ensure that the Rabinowitz action functional is Morse-Bott. This is precisely why we introduced condition (H4) - for the compleatness of the theory, more precisely to be able to apply Theorem 5 and Corollary 2.28 and thus ensure that the (MB) property is satisfied generically in an affine set of compact perturbations. Nevertheless, we claim that one could try and relax condition (H4), by replacing it with a weaker condition. Similarly as condition (H4) assures that condition (PO+)

persists under compact perturbations, the same way we would have to ensure that condition (PO) persists under compact perturbations. This may be not enough to assure the Morse-Bott property for generic perturbations, but should be enough to prove that for a generic perturbation the associated Rabinowitz action functional is Morse-Bott within a given action window. This would enable us to create a sequence of "truncated" Rabinowitz Floer homologies by restricting the homology to a fixed action window. By assuming that the property (PO) persists under compact perturbations, we should be able to repeat the argument from Section 3.9 and obtain that the "truncated" Rabinowitz Floer homology is invariant under compact perturbations. In the end it should allow us to define Rabinowitz Floer homology as a limit of "truncated" homologies similarly as in [13].

3.2 Openness under compact perturbations

In this section we will prove not only that the set of tentacular Hamiltonians is open under compact perturbations, but that around every tentacular Hamiltonian we can find an open neighborhood with satisfies all the properties of Definition 3.1 with one set of parameters chosen uniformly for the whole neighborhood.

Lemma 3.2. *Let $H_0 : \mathbb{R}^{2n} \to \mathbb{R}$ be a Hamiltonian satisfying Properties (H1), (H2) and (H3). Then for any compact set $K \subseteq \mathbb{R}^{2n}$, $K \neq \emptyset$, there exists an open subset $\mathscr{O}(H_0) \subseteq C_0^\infty(K)$, such that all the Hamiltonians from the set*

$$H_0 + \mathscr{O}(H_0) := \{H_0 + h \mid h \in \mathscr{O}(H_0)\}$$

satisfy Properties (H1), (H2) and (H3) and the set of parameter can be chosen uniformly for the whole set $H_0 + \mathscr{O}(H_0)$.

Proof. Let $c_1^0, c_2^0, c_3^0, c_4^0, c_5^0, \nu^0$ be the constants and $X_0^\dagger, X_0^\ddagger$ be the Liouville vector fields corresponding to H_0 assured by the fact that $H_0 \in \mathscr{H}$. Let $h \in C_0^\infty(K)$. Then from Property (H1) of H_0 one has

$$\begin{aligned} d(H_0+h)_x(X_0^\dagger) &= d(H_0)_x(X_0^\dagger) + dh_x(X_0^\dagger), \\ &\geq c_2^0\|x\|^2 - c_3^0 - \|h\|_{C^1(K)} \sup_{x\in K} \|X_0^\dagger\|, \\ &\geq c_2^0\|x\|^2 - c_3^0 - \|h\|_{C^1(K)} c_1^0 (\sup_{x\in K}\|x\| + 1) \end{aligned}$$

On the other hand, from Property (H2) of H_0 one obtains

$$\sup_{x\in\mathbb{R}^{2n}} \|D^3(H_0+h)\| \, \|x\| \leq \|h\|_{C^3(K)} \sup_{x\in K}\|x\| + \sup_{x\in\mathbb{R}^{2n}} \|D^3(H_0)\| \, \|x\|.$$

Now let us analyze Property (H3). Let c_0^4, c_0^5, ν_0 be the constants and $X_0^\ddagger$ is the Liouville vector field associated to H_0 by Property (H3).

Observe that for every $x \in \mathbb{R}^{2n}$ and $h \in C_0^\infty(K)$ one has

$$\begin{aligned} |H_0(x)| &\leq |(H_0+h)(x)| + |h(x)| \\ &\leq |(H_0+h)(x)| + \|h\|_{C^0(K)}. \end{aligned}$$

That means that whenever $\|h\|_{C^0(K)} < \frac{1}{2}\nu_0$, then we have the following inclusion

$$(H_0+h)^{-1}\Big(-\frac{1}{2}\nu_0, \frac{1}{2}\nu_0\Big) \subseteq H_0^{-1}(-\nu_0,\nu_0).$$

By property (H3) we have that for $x \in (H_0+h)^{-1}(-\frac{1}{2}\nu_0, \frac{1}{2}\nu_0)$ we have

$$\begin{aligned} d(H_0+h)_x(X_0^\ddagger) &\geq d(H_0)_x(X_0^\ddagger) + dh_x(X_0^\ddagger) \\ &\geq d(H_0)_x(X_0^\ddagger) + dh_x(X_0^\ddagger) \\ &\geq \inf_{H_0^{-1}((-\nu_0,\nu_0))} d(H_0)_x(X^\ddagger) - \|h\|_{C^1(K)} \sup_{K\cap H_0^{-1}(-\nu_0,\nu_0)} \|X_0^\ddagger\| \\ &= c_5^0 - c_0^4\|h\|_{C^1(K)}(\sup_K \|x\|^2+1). \end{aligned}$$

Now if we denote

$$\theta := \frac{1}{2}\min\Big\{\nu_0, \frac{c_0^5}{c_0^4(\sup_K\|x\|^2+1)}\Big\}, \tag{3.5}$$

and

$$\mathscr{O}(H_0) := \{h \in C_0^\infty \mid \|h\|_{C^3(K)} < \theta\}$$

then for all $h \in \mathscr{O}(H_0)$, we will have

$$d(H_0+h)_x(X_0^\dagger) \geq c_2^0\|x\|^2 - c_3^0 - \theta\sup_{x\in K}\|X_0^\dagger\| \qquad \forall x \in \mathbb{R}^{2n},$$

$$\sup_{x\in\mathbb{R}^{2n}} \|D^3(H_0+h)\|\,\|x\| \leq \theta\sup_{x\in K}\|x\| + \sup_{x\in\mathbb{R}^{2n}}\|D^3(H_0)\|\,\|x\|,$$

and for all $x \in (H_0+h)^{-1}(-\frac{1}{2}\nu_0, \frac{1}{2}\nu_0)$ we have

$$d(H_0+h)_x(X_0^\ddagger) \geq c_5^0 - c_0^4\|h\|_{C^1(K)}(\sup_K\|x\|^2+1) \geq \frac{1}{2}c_5^0 > 0.$$

In particular, we can choose the parameters uniformly for the whole family. In other words, we have the inclusion

$$H_0 + \mathscr{O}(H_0) \subseteq \mathscr{H},$$

and for all the Hamiltonians from this set the properties (H1), (H2), (H3) and (H4) are satisfied with parameters, which depend only on H_0 and K

$$c_1 := c_1^0 \quad c_2 := c_2^0 \quad c_3 := c_3^0 + \theta c_1^0(\sup_{x\in K}\|x\|+1)$$

$$c_4 := c_0^4 \quad c_5 := \tfrac{1}{2}c_0^5 \quad \nu := \frac{1}{2}\nu_0,$$

$$L := \theta\sup_{x\in K}\|x\| + \sup_{x\in\mathbb{R}^{2n}}\|D^3(H_0)\|\,\|x\|.$$

Moreover, for all $H \in H_0 + \mathscr{O}(H_0)$ we have uniform quadratic behavior:

$$\begin{aligned} \forall\, x \in \mathbb{R}^{2n} \qquad \|Hess_x H\| &\leq M, \\ \|\nabla H(x)\| &\leq h_1 + M\|x\|, \\ |H(x)| &\leq h_0 + h_1\|x\| + \frac{1}{2}M\|x\|^2, \end{aligned}$$

$$\text{where} \quad M := \theta + \|Hess_0 H_0\| + L, \quad h_0 := \theta + |H_0(0)|, \quad h_1 := \theta + \|\nabla H_0(0)\|.$$

□

Now if we combine the above lemma with Corollary 2.28, we obtain that the set of tentacular Hamiltonians is open under compact perturbations

Corollary 3.3. *Fix $H \in \mathscr{H}$ and a compact set $K \subseteq \mathbb{R}^{2n}$. If $\mathscr{O}(H)$ is the open set associated to H and K as in Lemma 3.2, then the Hamiltonians from the affine set $H + \mathscr{O}(H)$ are tentacular:*

$$H + \mathscr{O}(H) \subseteq \mathscr{H}.$$

3.3 Bounds on the moduli spaces

The main part of the construction of the Rabinowitz Floer homology is to establish uniform bounds on the moduli spaces of Floer trajectories. Since we would like also to assure that the defined homology is invariant under small perturbations, we will aim to prove the bounds in more generality namely for a homotopy of Hamiltonians and almost complex structures. This Section presents an overview of the proof of bounds on the moduli spaces of Floer trajectories, which occupies Sections 3.4 to 3.8.

However, rather than stating the theorem for the family of tentacular Hamiltonians, we choose to formulate it in more generality omitting Property (H4) and assuming (PO) instead. Recall the fact that (H4) implies (PO) has been proven in Corollary 2.28, equation (2.42). This way, formulating Theorem 7. in terms of Property (PO), rather than (H4), we obtain a stronger statement, which can be applied to systems which satisfy (PO), but not (H4). An example of such a system is the Hamiltonian for the restricted 3-body problem. A possible application would be to construct a "truncated" Rabinowitz Floer homology, i.e. to define a homology for an action window and analyze its limit in a similar way it has been done for the compact hypersurfaces in [13], Chapter 3. However, in this thesis we restrict ourselves to the case of Hamiltonians satisfying Definition 3.1. in order to achieve completeness of the argument, in particular to assure that Property (PO+) persists under compact perturbations.

Another fact worth mentioning is that Property (SG) does not play a major part in the proof and is necessary only to achieve boundedness for the moduli spaces between $(\mathcal{A}^{H_0})^{-1}(0)$ and $(\mathcal{A}^{H_1})^{-1}(0)$ in the case of a family of Hamiltonians. In particular, if we are interested in obtaining the bounds for a fixed Hamiltonian, then the assertions of the Theorem 7. still holds true without assumption (SG) for a constant family.

To formulate Theorem 7 let us recall the definition of the set $\mathscr{C}(\mathcal{A}^H, N) \subseteq Crit(\mathcal{A}^H)$ formulated in (2.6) for every compact subset $N \subseteq H^{-1}(0)$ and Definition 2.2) of a moduli space of Floer trajectories for a homotopy of Hamiltonians and almost complex structures $\Gamma = \{H_s, J_s\}_{s\in\mathbb{R}}$, which satisfies (2.4) formulated in subsection 2.1.2. Next, let us observe that for a fixed Hamiltonian $H_0 : \mathbb{R}^{2n} \to \mathbb{R}$ satisfying Properties (H1), (H2) and (H3) a compact set $K \subseteq \mathbb{R}^{2n}$, $K \neq \emptyset$ by Lemma 3.2 there exists and open set $\mathscr{O}(H_0)$, such that all the Hamiltonians from $H_0 + \mathscr{O}(H_0)$ also satisfy Properties (H1), (H2) and (H3) and the set of parameters can be chosen uniformly for the whole family $H_0 + \mathscr{O}(H_0)$. As a result, the constants $\tilde{c}$ and ε_0 calculated in Lemma 3.5 can also be chosen uniformly for the whole family $H_0 + \mathscr{O}(H_0)$.

Theorem 7. *Fix a Hamiltonian $H_0 : \mathbb{R}^{2n} \to \mathbb{R}$ satisfying Properties (H1), (H2), (H3), (PO) and (SG), a compact set $K \subseteq \mathbb{R}^{2n}$, $K \neq \emptyset$, an open, precompact subset $\mathcal{V} \subseteq \mathbb{R}^{2n}$ and a constant $\eta > 0$. Let $\mathscr{O}(H_0)$ be an open set associated to H_0 and K as in Lemma 3.2 and let $\Gamma = \{H_s, J_s\}_{s\in\mathbb{R}}$ be a smooth homotopy of Hamiltonians and almost complex structures, constant outside of $[0,1]$ (i.e. Γ satisfies (2.4)) and such that for all $s \in \mathbb{R}$*

$$H_s - H_0 \in \mathscr{O}(H_0)$$
$$J_s = \{J_{t,s}(\cdot,\eta)\}_{S^1\times\mathbb{R}} \in \mathscr{J}\Big(\mathbb{R}^{2n}, \omega_0, \mathcal{V} \times \big((-\infty,\eta) \cup (\eta,\infty)\big)\Big).$$

Let $\tilde{c} < \infty$ and $\varepsilon_0 > 0$ be constants as in Lemma 3.5 associated to the set $H_0 + \mathscr{O}(H_0)$ and assume that the family homotopy Γ satisfies

$$\Big(\tilde{c} + \frac{1}{\varepsilon_0}\max\{\|J\|_\infty^3, 1\}\Big)\|\partial_s H_s\|_\infty < \frac{1}{8}. \tag{3.6}$$

If we now fix $a, b \in \mathbb{R}$ and a compact subset $N \subseteq H_0^{-1}(0)$, then for each pair (Λ_0, Λ_1) of connected components

$$\Lambda_0 \subseteq \mathscr{C}(\mathcal{A}^{H_0}, N) \cap (\mathcal{A}^{H_0})^{-1}([a,\infty)) \tag{3.7}$$
$$\Lambda_1 \subseteq Crit(\mathcal{A}^{H_1}) \cap (\mathcal{A}^{H_1})^{-1}((-\infty,b]) \tag{3.8}$$

the associated moduli space $\mathscr{M}^\Gamma(\Lambda_0,\Lambda_1) \subseteq C^\infty(\mathbb{R}\times S^1, \mathbb{R}^{2n}\times\mathbb{R})$ is contained in a compact set in $\mathbb{R}^{2n+1}$.

Note that if we take Γ to be a constant homotopy, i.e $(H_s, J_s) = (H, J)$ for all $s \in \mathbb{R}$, where H satisfies conditions (PO), (SG), (H1), (H2) and (H3), whereas $J \in \mathscr{J}\Big(\mathbb{R}^{2n}, \omega_0, \mathcal{V} \times \big((-\infty,\eta) \cup (\eta,\infty)\big)\Big)$, then condition (3.6) is automatically satisfied and the assertions of the theorem hold true.

Proof. We will here present the general outline of the proof with references to the propositions and lemmas where the specific statements are proven.

The firsts step is to prove global bounds on the action, energy and η parameter for the whole moduli space, which is explained in Section 3.4. Secondly, we will explain how to divide the space into three sets and bound the $L^2 \times \mathbb{R}$ norm on each set differently, which is the subject of in Sections 3.5 and 3.6. Afterwards, we will try to localize the Floer trajectory and analyze how the it oscillates between those sets, which is explained in Section 3.7. Finally, in Section 3.8, we will establish the $L^\infty \times \mathbb{R}$ bounds using the maximum principle.

Let us first establish the bounds on the action, energy and η parameter. In other words, we want to prove that if we fix a pair of connected components Λ_0, Λ_1 as in (3.7) and (3.8) then the corresponding $\mathscr{M}^\Gamma(\Lambda_0, \Lambda_1)$ has a global bound on the action, energy and η parameter. In the case of a constant family of Hamiltonians $H_s = H$ the action and the energy of $u \in \mathscr{M}^\Gamma(\Lambda_0, \Lambda_1)$ are trivially bounded namely

$$\frac{d}{ds}(\mathcal{A}^H \circ u)(s) = \|\partial_s u(s)\|^2 \geq 0,$$

and therefore we have the relation

$$\begin{aligned} \int_{-\infty}^{\infty} \|\nabla \mathcal{A}^{H_s}(u(s))\|^2 = \int_{-\infty}^{\infty} \|\partial_s u(s)\|^2 ds \leq b - a, \\ \text{and} \qquad a \leq \mathcal{A}^H(u(s)) \leq b. \end{aligned}$$

However, in the case of a non-constant family $\{H_s\}_{s\in\mathbb{R}}$

$$\frac{d}{ds}(\mathcal{A}^{H_s} \circ u)(s) = \|\partial_s u(s)\|^2 + \eta(s) \int_0^1 \partial_s H_s(v(s,t))dt,$$

which makes it more difficult to prove the bounds on the energy and the action. Nevertheless, we have obtained the bounds by following the approach presented in [12], i.e. by first proving in Lemma 3.5 that all Hamiltonians in $H_0 + \mathscr{O}(H_0)$ satisfy a linearity condition between the action and the η parameter

$$|\eta| \leq \tilde{c}(|\mathcal{A}^H(v,\eta)| + 1),$$

and then in Proposition 3.4 obtaining the bounds on the action, energy and also η parameter uniform for the whole $\mathscr{M}^\Gamma(\Lambda_0, \Lambda_1)$ provided the homotopy $\Gamma = \{(H_s, J_s)\}_{s\in\mathbb{R}}$ satisfies inequality (3.6)

$$\Big(\tilde{c} + \frac{1}{\varepsilon_0}\max\{\|J\|_\infty^3, 1\}\Big)\|\partial_s H_s\|_\infty \leq \frac{1}{8}.$$

Now, by Proposition 3.4, there exist constants $\mathfrak{a}, \mathfrak{y}, \mathfrak{e} > 0$, such that for every $u \in \mathscr{M}^\Gamma(\Lambda_0, \Lambda_1)$, $u(s) = (v(s), \eta(s))$ we have

$$\begin{aligned} |\eta(s)| \leq \mathfrak{y} \quad \& \quad |\mathcal{A}^{H_s}(u(s))| \leq \mathfrak{a} \quad \forall\, s \in \mathbb{R} \\ \text{and} \qquad \int_{-\infty}^{\infty} \|\nabla \mathcal{A}^{H_s}(u(s))\|^2 ds \leq \mathfrak{e}. \end{aligned}$$

Note that $\mathfrak{y}$ and $\mathfrak{e}$ depend continuously on $\|J\|_\infty$, whereas $\mathfrak{a}$ doesn't depend on $\|J\|_\infty$. In particular, recalling the definition of the set of infinitesimal action derivation (Definition 3.7), we have

$$u \in \mathscr{M}^\Gamma(\Lambda_0, \Lambda_1) \quad \& \quad \|\nabla^{J_s}\mathcal{A}^{H_s}(u(s))\| < \varepsilon \leq \varepsilon_0 \quad \Longrightarrow \quad u(s) \in \mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon).$$

The set of infinitesimal action derivation will therefore play an important role in our proof and the first step will be to localize it.

Take $\delta > 0$, then by Proposition 3.8 for

$$u \in \mathscr{M}^{\Gamma}(\Lambda_0, \Lambda_1) \quad \& \quad \|\nabla^{J_s}\mathcal{A}^{H_s}(u(s))\| < \varepsilon \leq \varepsilon_2(\delta, \|J\|_\infty),$$

we have either

$$\|v(s)\|_{L^2} \geq v_2(\delta, \sup_{x\in K\cup\mathcal{V}} \|x\|) \quad \& \quad dist_{W^{1,2}\times\mathbb{R}}(u(s), \Sigma) \leq \delta$$

or $\|u(s)\|_{W^{1,2}\times\mathbb{R}}$ is uniformly bounded. In other words, if we denote by $\Sigma \subseteq C^\infty(S^1, \mathbb{R}^{2n})$ the set of constant loops on $H_0^{-1}(0)$ and by $\mathcal{U}^1_\delta$ the $W^{1,2}\times\mathbb{R}$ neighborhood of $\Sigma\times\{0\}$ as in(3.22), then Proposition 3.8 states that for every $\delta > 0$ and $\varepsilon \in (0, \varepsilon_2(\delta, \|J\|_\infty))$ the corresponding set $K^1_\delta \subseteq C^\infty(S^1, \mathbb{R}^{2n})$ defined as in (3.24) is bounded in $W^{1,2}\times\mathbb{R}$ norm and satisfies

$$\mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon) \subseteq \mathcal{K}^1_\delta \cup \mathcal{U}^1_\delta.$$

By Lemma 3.9 homotopy Γ satisfies Novikov finiteness condition. As a result, by defining K^1_δ as in (3.24) we assured that

$$\mathscr{C}(\mathcal{A}^{H_0}, N) \cap (\mathcal{A}^{H_0})^{-1}([a, B(\Gamma, a, b)]) \subseteq \mathcal{K}^1_\delta,$$

where $B(\Gamma, a, b)$ defined as in (2.24). In particular, for we can assume that $\Lambda_0 \subseteq \mathcal{K}^1_\delta$.

Having localized $\mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon)$ we would now like to establish global L^2 bounds on $\mathscr{M}^{\Gamma}(\Lambda^-, \Lambda^+)$. However, to establish the L^2 bounds on the v component of a Floer trajectory one has to analyze the Floer trajectory not only outside of set of infinitesimal action derivation, but also how far it travels within $\mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon)$, along the hypersurface $\Sigma\times\{0\}$. To obtain the uniform bounds, we will first cover the space $C^\infty(S^1, \mathbb{R}^{2n})\times\mathbb{R}$ by three sets on which the Floer trajectory can be bounded differently.

Let $\tilde{\delta}(H_0)$ be as in Lemma 4.14 and fix

$$\delta \in \big(0, \min\big\{\mathfrak{y}, \frac{1}{3M}, \tilde{\delta}(H_0)\big\}\big). \tag{3.9}$$

That means that the δ-tubular neighborhood of $H^{-1}(0)$ is well defined and δ satisfies assumptions of Proposition 3.16. Then by Proposition 3.8 for every $\varepsilon \in (0, \varepsilon_2(\delta/2, \|J\|_\infty))$ we have the following partition of $\mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon)$

$$\mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon) \subseteq \mathcal{K}^1_{\delta/2} \cup \mathcal{U}^1_{\delta/2}.$$

The reason why we choose here $\frac{1}{2}\delta$ not δ is to estimate the number of oscillations of the Floer trajectory and will became apparent later in the proof of Lemma 3.17.

Now take $v_2(\delta/2, \max_{x\in K\cup\mathcal{V}}\|x\|)$ as in Proposition 3.8, v_3 as in Lemma 3.9 and r as in (3.28) and denote

$$v_4 = \max\Big\{r, v_3, \sup_{x\in K}\|x\|, v_2(\delta/2, \max_{x\in K\cup\mathcal{V}}\|x\|)\Big\}. \tag{3.10}$$

Observe that v_4 does not depend on Γ, but only on the parameters chosen uniformly for the whole set $\mathscr{O}(H_0)$ and on the fact that Γ satisfies (3.6). Now, in terms of v_4 we can define the following subset of $C^\infty(S^1, \mathbb{R}^{2n}) \times \mathbb{R}$:

$$\mathcal{K}^0_\delta := \left\{ \left. (v, \eta) \in C^\infty(S^1, \mathbb{R}^{2n}) \times \mathbb{R} \;\right|\; \begin{array}{c} |\eta| \leq \mathfrak{y} \\ \|v\|_{L^2} \leq v_4 + \delta \end{array} \right\} \tag{3.11}$$

Such defined $\mathcal{K}^0_\delta$ is bounded in $L^2 \times \mathbb{R}$ norm. By definition of $\mathcal{K}^1_{\delta/2}$ and $\mathcal{K}^0_\delta$ we know that every Floer trajectory starts in these sets due to the inclusion

$$\mathscr{C}(\mathcal{A}^{H_0}, N) \cap (\mathcal{A}^{H_0})^{-1}([a, B(\Gamma, a, b)]) \subseteq \mathcal{K}^0_{\delta/2} \subseteq \mathcal{K}^0_\delta.$$

Now let $\mathscr{N}^\delta_{v_4}$ be the set as in (3.27). Then we have the inclusions

$$\begin{aligned} \mathcal{U}^0_\delta \setminus \mathcal{K}^0_\delta \;&\subseteq\; \mathscr{N}^\delta_{v_4}, \\ \mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon) \subseteq \mathcal{K}^0_\delta \cup \mathcal{U}^0_\delta &= \mathcal{K}^0_\delta \cup \mathscr{N}^\delta_{v_4}. \end{aligned}$$

Note that our choice of $\delta <$$\mathfrak{y}$and

$$J_s \in \mathscr{J}\Big(\mathbb{R}^{2n}, \omega_0, \mathcal{V} \times \big((-\infty, \mathfrak{y}) \cup (\mathfrak{y}, \infty)\big)\Big)$$

implies that

$$J_s\Big|_{\mathcal{U}^0_\delta} = J_0,$$

where J_0 is the standard almost complex structure. Moreover, observe that all H_s differ only outside a compact set namely $H_s - H_0 \in C^\infty_0(K)$. In particular the definition of $\mathscr{N}^\delta_{v_4}$ is independent of which H_s we choose, since we have taken

$$v_4 \geq \sup_{x \in K} \|x\|.$$

That means that if we assume $s \notin (0, 1)$ then on $\mathscr{N}^\delta$ the pair (H_s, J_s) is constant equal either (H_0, J_0) or (H_1, J_0). Finally, we have assumed $v_4 \geq r$, where r is as in (3.28) and thus we can apply Proposition 3.16 directly.

This way we can cover the space $C^\infty(S^1, \mathbb{R}^{2n}) \times \mathbb{R}$ by three sets

$$\mathcal{K}^0_\delta, \qquad \mathscr{N}^\delta_r, \qquad (C^\infty(S^1, \mathbb{R}^{2n}) \times \mathbb{R}) \setminus \mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon),$$

on which the Floer trajectory can be bounded differently. Whenever the Floer trajectory is outside $\mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon)$, then its growth in the $L^2 \times \mathbb{R}$ norm can be estimated using results from Lemma 3.17. On the other hand, Proposition 3.16 gives the bounds on the growth of the Floer trajectory in the $L^2 \times \mathbb{R}$ norm, provided the Floer trajectory is in $\mathscr{N}^\delta_r$. To obtain the uniform bounds for the whole Floer trajectory, we will also have to determine how it oscillates between the two non-compact sets, i.e. between $\mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon) \setminus \mathcal{K}^0_\delta$ and $(C^\infty(S^1, \mathbb{R}^{2n}) \times \mathbb{R}) \setminus \mathscr{N}^\delta_{v_4}$, which is carried out in Lemma 3.17.

Having obtained the bound on the number of oscillation of a Floer trajectory, it gives us sufficient means to infer the global bounds on $\mathscr{M}^\Gamma(\Lambda_0, \Lambda_1)$ in the $L^2 \times \mathbb{R}$, which are proven in Proposition 3.18. As a consequence for every $\varepsilon < \varepsilon_2(\delta, \|J\|_\infty)$ the set

$$\bigcup_{u \in \mathscr{M}^\Gamma(\Lambda^0,\Lambda^1)} (u(\mathbb{R}) \cap \mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon))$$

is uniformly bounded in the $W^{1,2}(S^1) \times \mathbb{R}$ norm. This enables us to choose a compact subset $\mathcal{K}^\infty \subseteq \mathbb{R}^{2n}$, such that

$$\mathcal{V} \cup K \subseteq \mathcal{K}^\infty, \tag{3.12}$$

$$\forall\ (v, \eta) \in \bigcup_{u \in \mathscr{M}^\Gamma(\Lambda^0,\Lambda^1)} (u(\mathbb{R}) \cap \mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon)) \qquad v(S^1) \subseteq \mathcal{K}^\infty. \tag{3.13}$$

Now, if we choose $u \in \mathscr{M}^\Gamma(\Lambda_0, \Lambda_1), u(s,t) = (v(s,t), \eta(s))$ and take a connected component

$$\Omega \subseteq v^{-1}(\mathbb{R}^{2n} \setminus \mathcal{K}^\infty)$$

then by (3.13) and (3.12) the following holds for all $(s,t) \in \Omega$

$$\begin{aligned} H^s(v(s,t)) = H^0(v(s,t)) &= H^1(v(s,t)) \\ J_t(v(s,t), \eta(s)) &= J_0 \\ \|\nabla \mathcal{A}^H(u(s))\|_{L^2 \times \mathbb{R}} &\geq \varepsilon \end{aligned}$$

and therefore we can apply Proposition 3.20 directly, which together with Asonov's maximum principle assures that there exists a constant $C(\mathfrak{e}, \mathfrak{y}, \mathbf{v}, \varepsilon) > 0$, such that the following inequality is satisfied

$$\sup_\Omega \|v(s,t)\| \leq \sup_{\partial\mathcal{K}^\infty} \|v(s,t)\| + C(\mathfrak{e}, \mathfrak{y}, \mathbf{v}, \varepsilon).$$

By Proposition 3.20 the constant $C(\mathfrak{e}, \mathfrak{y}, \mathbf{v}, \varepsilon) > 0$ does not depend on the choice of $u \in \mathscr{M}^\Gamma(\Lambda_0, \Lambda_1)$ or Ω, therefore, we can conclude that for all $u \in \mathscr{M}^\Gamma(\Lambda_0, \Lambda_1)$ we have

$$\sup_{s\in\mathbb{R}} \|u(s)\|_{L^\infty(S^1)\times\mathbb{R}} \leq \sup_{x \in \mathcal{K}^\infty} \|x\| + \mathfrak{y} + C(\mathfrak{e}, \mathfrak{y}, \mathbf{v}, \varepsilon),$$

thus establishing uniform $L^\infty \times \mathbb{R}$ bounds on $\mathscr{M}^\Gamma(\Lambda_0, \Lambda_1)$. □

3.4 Bounds on the action

Proposition 3.4. *Fix $H_0 \in C^\infty(\mathbb{R}^{2n})$ satisfying properties (H1), (H2) and (H3) and a compact set $K \subseteq \mathbb{R}^{2n}$, $K \neq \emptyset$ and let $\mathscr{O}(H_0) \subseteq C_0^\infty(K)$ be the open set as in Lemma 3.2. Let $\Gamma = \{H_s, J_s\}_{s\in\mathbb{R}}$ be a smooth homotopy of Hamiltonians and almost complex structures, constant outside $[0,1]$, such that*

$$\forall\ s \in \mathbb{R} \qquad (H_s, J_s) \in (H_0 + \mathscr{O}(H_0)) \times \mathscr{J}(\mathbb{R}^{2n}, \omega_0)$$

and Γ satisfies inequality (3.6).

Let $u \in C^\infty(\mathbb{R}, C^\infty(S^1, \mathbb{R}^{2n}) \times \mathbb{R})$ be a Floer trajectory of the corresponding time-dependent action functionals $\mathcal{A}^{H_s}$, such that

$$\lim_{s\to-\infty} \mathcal{A}^{H_s}(u(s)) \geq a \quad \text{and} \quad \lim_{s\to\infty} \mathcal{A}^{H_s}(u(s)) \leq b.$$

Then $\|\eta\|_{L^\infty}$, $\|\mathcal{A}^{H_s}(u(s))\|_{L^\infty}$ and $\|\nabla^{J_s}\mathcal{A}^{H_s}(u)\|_{L^2}$ are uniformly bounded by constants which depend only on a, b the set of Hamiltonians $H_0 + \mathscr{O}(H_0)$ and continuously on $\|J\|_\infty = \max_{s\in[0,1]} \|J_s\|_\infty$.

Proof. Moreover, since $J_s \in \mathscr{J}(\mathbb{R}^{2n}, \omega_0)$ (for definition see (2.1)) and is constant outside of $[0,1]$ as a consequence we have a limit on

$$\|J\|_\infty = \max_{s\in[0,1]} \|J_s\|_\infty < +\infty$$

and thus for every $s \in \mathbb{R}$

$$\frac{1}{\max\{\|J\|_\infty, 1\}} \|\cdot\|_{L^2\times\mathbb{R}} \leq \|\cdot\|^2_{g_{J_s}} \leq \max\{\|J\|_\infty, 1\}\|\cdot\|_{L^2\times\mathbb{R}}$$

Analogically, for every $s \in \mathbb{R}$ we have

$$\frac{\|\nabla\mathcal{A}^{H_s}(u(s))\|_{L^2\times\mathbb{R}}}{\max\{\|J\|_\infty, 1\}} \leq \|\nabla^{J_s}\mathcal{A}^{H_s}(u(s))\|_{L^2\times\mathbb{R}} \leq \max\{\|J\|_\infty, 1\}\|\|\nabla\mathcal{A}^{H_s}(u(s))\|_{L^2\times\mathbb{R}}.$$

Therefore

$$\begin{aligned}
\frac{\|\nabla\mathcal{A}^{H_s}(u)\|_{L^2}}{\max\{\|J\|^2_\infty, 1\}} &\leq \frac{\|\nabla^{J_s}\mathcal{A}^{H_s}(u)\|_{L^2}}{\max\{\|J\|_\infty, 1\}} \\
&\leq \Big(\int_{-\infty}^{\infty} \|\nabla^{J_s}\mathcal{A}^{H_s}(u(s))\|^2_{g_{J_s}} ds\Big)^{\frac{1}{2}} \\
&\leq (\max\{\|J\|_\infty, 1\})\|\nabla^{J_s}\mathcal{A}^{H_s}(u)\|_{L^2} \\
&\leq \max\{\|J\|^2_\infty, 1\}\|\nabla\mathcal{A}^{H_s}(u)\|_{L^2} \qquad (3.14)
\end{aligned}$$

Since $u \in C^\infty(\mathbb{R}, C^\infty(S^1, \mathbb{R}^{2n}) \times \mathbb{R})$ is a Floer trajectory one can calculate the derivative of the action functional over u:

$$\begin{aligned}
\frac{d}{ds}\mathcal{A}^{H_s}(u(s)) &= \|\nabla^{J_s}\mathcal{A}^{H_s}(u(s))\|^2_{g_{J_s}} + (\partial_s\mathcal{A}^{H_s})(u(s)) \\
&= \|\nabla^{J_s}\mathcal{A}^{H_s}(u(s))\|^2_{g_{J_s}} + \eta(s)\int_0^1 \partial_s H_s(v(s,t))dt
\end{aligned}$$

Using the assumptions on the action of the endpoints one obtains

$$\begin{aligned}
\mathcal{A}^{H_{s_0}}(u(s_0)) &= \lim_{s\to-\infty} \mathcal{A}^{H_0}(u(s)) + \int_{-\infty}^{s_0} \frac{d}{ds}\mathcal{A}^{H_s}(u(s))ds \\
&\geq a + \int_{-\infty}^{s_0} \|\nabla^{J_s}\mathcal{A}^{H_s}(u(s))\|^2_{g_{J_s}} ds + \int_0^{s_0} \eta(s)\int_0^1 \partial_s H_s(v(s,t))dtds, \\
&\geq a - \|\eta\|_\infty\|\partial_s H_s\|_\infty.
\end{aligned}$$

Analogically, one obtains

$$\begin{aligned}\mathcal{A}^{H_{s_0}}(u(s_0)) &\leq b - \int_{s_0}^{\infty} \|\nabla^{J_s}\mathcal{A}^{H_s}(u(s))\|^2_{g_{J_s}} ds - \int_{s_0}^{1} \eta(s) \int_0^1 \partial_s H_s(v(s,t))dtds \\ &\leq b + \|\eta\|_\infty \|\partial_s H_s\|_\infty,\end{aligned}$$

which together leads to

$$\|\mathcal{A}^{H_s}(u)\|_{L^\infty} \leq \max\{|a|,|b|\} + \|\eta\|_\infty \|\partial_s H_s\|_\infty. \tag{3.15}$$

Analogically, using (3.14) we can bound the energy

$$\begin{aligned}b - a &\geq \int_{-\infty}^{\infty} \frac{d}{ds}\mathcal{A}^{H_s}(u(s))ds \\ &= \int_{-\infty}^{\infty} \|\nabla^{J_s}\mathcal{A}^{H_s}(u(s))\|^2_{g_{J_s}} ds + \int_0^1 \eta(s) \int_0^1 \partial_s H_s(v(s,t))dtds\end{aligned}$$

$$\|\nabla^{J_s}\mathcal{A}^{H_s}(u)\|^2_{L^2} \leq \max\{\|J\|^2_\infty, 1\}(b - a + \|\eta\|_\infty\|\partial_s H_s\|_\infty) \tag{3.16}$$

In particular the convergence of the integral

$$\int_{-\infty}^{\infty} \|\nabla\mathcal{A}^{H_s}(u(s))\|^2 ds \leq \max\{\|J\|_\infty, 1\} \int_{-\infty}^{\infty} \|\nabla^{J_s}\mathcal{A}^{H_s}(u(s))\|^2 ds$$

implies that if we fix $\varepsilon_0 > 0$ as in Lemma 3.5, then for small enough s

$$\|\nabla\mathcal{A}^{H_s}(u(s))\|_{L^2} < \varepsilon_0.$$

This assures that for all $s \in \mathbb{R}$ the following value $\tau_0(s)$ is well defined and finite

$$\tau_0(s) := \inf\{\tau \leq s \mid \|\nabla\mathcal{A}^{H_\tau}(u(\tau))\| < \varepsilon_0\}.$$

For $\tau \in [\tau_0(s), s]$ we have

$$\|\nabla\mathcal{A}^{H_\tau}(u(\tau))\| \geq \varepsilon_0.$$

therefore by (3.16)

$$\begin{aligned}|s - \tau_0(s)|\varepsilon_0^2 &\leq \int_s^{\tau_0(s)} \|\nabla\mathcal{A}^{H_\tau}(u(\tau))\|^2 d\tau \\ &\leq \|\nabla\mathcal{A}^{H_s}(u)\|^2_{L^2} \\ &\leq (\max\{\|J\|_\infty, 1\})^4(b - a + \|\eta\|_\infty\|\partial_s H_s\|_\infty), \\ |s - \tau_0(s)| &\leq \frac{(\max\{\|J\|_\infty, 1\})^4}{\varepsilon_0^2}(b - a + \|\eta\|_\infty\|\partial_s H_s\|_\infty).\end{aligned}$$

Now using the above bounds we can calculate

$$\begin{aligned}
|\eta(s)-\eta(\tau_0(s))| &\leq \int_{\tau_0(s)}^{s} |\partial_s\eta(\tau)|d\tau \\
&\leq \sqrt{|s-\tau_0(s)|}\sqrt{\int_{\tau_0(s)}^{s} |\partial_s\eta(\tau)|^2 d\tau} \\
&\leq \sqrt{|s-\tau_0(s)|}\sqrt{\int_{\tau_0(s)}^{s} \|\nabla^{J_\tau}\mathcal{A}^{H_\tau}(u(\tau))\|^2 d\tau} \\
&\leq \frac{\max\{\|J\|_\infty^3, 1\}}{\varepsilon_0}(b-a+\|\eta\|_\infty\|\partial_s H_s\|_\infty). \qquad (3.17)
\end{aligned}$$

On the other hand, by definition of $\tau_0(s)$ the result of Lemma 3.5 applies giving

$$|\eta(\tau_0(s))| \leq \tilde{c}(|\mathcal{A}^{H_{\tau_0(s)}}(u(\tau_0(s)))|+1)$$

Now combining it with (3.15) and (3.17) we can estimate that for any $s \in \mathbb{R}$ we have

$$\begin{aligned}
|\eta(s)| &\leq |\eta(\tau_0(s))| + |\eta(s)-\eta(\tau_0(s))| \\
&\leq \tilde{c}(|\mathcal{A}^{H_{\tau_0(s)}}(u(\tau_0(s)))|+1) + \frac{\max\{\|J\|_\infty^3, 1\}}{\varepsilon_0}(b-a+\|\eta\|_\infty\|\partial_s H_s\|_\infty) \\
&\leq \tilde{c}(\max\{|a|,|b|\} + \|\eta\|_\infty\|\partial_s H_s\|_\infty + 1) + \frac{1}{\varepsilon_0}(b-a+\|\eta\|_\infty\|\partial_s H_s\|_\infty) \\
&= \tilde{c}(\max\{|a|,|b|\}+1) + \frac{(b-a)}{\varepsilon_0}\max\{\|J\|_\infty^3, 1\} \\
&\qquad + \Big(\tilde{c} + \frac{1}{\varepsilon_0}\max\{\|J\|_\infty^3, 1\}\Big)\|\eta\|_\infty\|\partial_s H_s\|_\infty
\end{aligned}$$

Since the above inequality holds for every $s \in \mathbb{R}$, therefore in fact we have

$$\|\eta\|_\infty \leq \tilde{c}(\max\{|a|,|b|\}+1)+\frac{(b-a)}{\varepsilon_0}\max\{\|J\|_\infty^3, 1\}+\Big(\tilde{c}+\frac{1}{\varepsilon_0}\max\{\|J\|_\infty^3, 1\}\Big)\|\eta\|_\infty\|\partial_s H_s\|_\infty,$$

which together with (3.6), (3.15) and (3.16) gives the claimed bounds

$$\begin{aligned}
\|\eta\|_\infty &\leq \frac{8}{7}\Big(\tilde{c}(\max\{|a|,|b|\}+1) + \frac{b-a}{\varepsilon_0}\max\{\|J\|_\infty^3, 1\}\Big), \qquad (3.18) \\
\|\eta\|_\infty\|\partial_s H_s\|_\infty &\leq \frac{8}{7}\Big(\tilde{c}(\max\{|a|,|b|\}+1) + \frac{b-a}{\varepsilon_0}\max\{\|J\|_\infty^3, 1\}\Big)\|\partial_s H_s\|_\infty \\
&\leq \frac{1}{7}(\max\{|a|,|b|\}+1+b-a) \\
\|\nabla^{J_s}\mathcal{A}^{H_s}(u(s))\|_{L^2}^2 &\leq \frac{1}{7}\max\{\|J\|_\infty^2, 1\}\Big(8(b-a)+\max\{|a|,|b|\}+1\Big) \qquad (3.19) \\
\|\mathcal{A}^{H_s}(u)\|_{L^\infty} &\leq \frac{1}{7}(8\max\{|a|,|b|\}+1+b-a) \qquad (3.20)
\end{aligned}$$

where $\tilde{c}$ and ε_0 are the constants associated to the set $H_0+\mathscr{O}(H_0)$ by Lemma 3.5. □

Observe that the bound on the action does not depend on the homotopy, i.e. is uniform for all homotopies satisfying the assumptions of Proposition 3.4. Moreover, the bounds on the η parameter and on the energy depend continuously on $\|J\|_\infty$, but not on the choice of $\{H_s\}_{s\in\mathbb{R}}$.

Now we will present the proofs of the lemmas used in the proof of the proposition above.

Lemma 3.5. *Fix $H_0 \in C^\infty(\mathbb{R}^{2n})$ satisfying properties (H1), (H2) and (H3) and a compact set $K \subseteq \mathbb{R}^{2n}$, $K \neq \emptyset$ and let $\mathscr{O}(H_0) \subseteq C_0^\infty(K)$ be the open set as in Lemma 3.2. Then there exists $\varepsilon_0 > 0$ and $\tilde{c} > 0$, such that for every $H \in H_0 + \mathscr{O}(H_0)$ whenever $(v,\eta) \in C^\infty(S^1,\mathbb{R}^{2n}) \times \mathbb{R}$ satisfies*

$$\|\nabla\mathcal{A}^H(v,\eta)\|_{L^2(S^1\times\mathbb{R})} < \varepsilon_0,$$

then

$$|\eta| \leq \tilde{c}(|\mathcal{A}^H(v,\eta)| + 1).$$

Proof. Take the set of parameters uniform for the whole set $\mathscr{O}(H_0)$ as in Lemma 3.2, then we will show that the assertion holds true for

$$\varepsilon_0 := \min\left\{\frac{c_5}{2}, \frac{\nu}{2}\min\left\{1, \frac{1}{M\mathrm{v}_0 + h_1}\right\}\right\}$$
$$\text{and} \quad \tilde{c} := \max\left\{\frac{2}{c_5}, c_4(\mathrm{v}_0^2 + 1)\right\},$$

where the constants are associated to $H_0 + \mathscr{O}(H_0)$ trough Lemma 3.2 and v_0 depends only on these constants and is defined in (3.21).

Step 1:
Recall that for every Liouville vector field we have

$$\mathcal{A}^H(v,\eta) - \mathcal{A}'^H_{(v,\eta)}(Y,\eta) = \eta\int dH(Y).$$

Since H satisfies Property (H1), we can apply the above equality to the Liouville vector field $X^\dagger$ to obtain

$$|\mathcal{A}^H(v,\eta)| + \|\nabla\mathcal{A}^H(v,\eta)\|(c_1(\|v\|_{L^2} + 1) + |\eta|) \geq |\eta|(c_2\|v\|^2_{L^2} - c_3).$$

Therefore whenever $\|\nabla\mathcal{A}^H(v,\eta)\| < \varepsilon_0 \leq \frac{c_5}{2}$ this implies

$$|\mathcal{A}^H(v,\eta)| + \frac{c_5}{2}(c_1(\|v\|_{L^2} + 1) + |\eta|) \geq |\eta|(c_2\|v\|^2_{L^2} - c_3),$$

and thus with $\|v\|_{L^2}$ large enough, that is

$$\|v\|_{L^2} > \sqrt{\frac{2c_3 + c_5}{2c_2}},$$

the following inequality holds

$$\frac{|\mathcal{A}^H(v,\eta)| + \frac{c_1 c_5}{2}(\|v\|_{L^2}+1)}{c_2\|v\|_{L^2}^2 - c_3 - \frac{c_5}{2}} \geq |\eta|.$$

Now take

$$\mathrm{v}_0 := \max\left\{\sqrt{\frac{c_3+c_5}{c_2}}, \frac{c_1c_5 + \sqrt{(c_1c_5)^2 + 8c_2c_4(c_1c_5 + 2c_3c_4 + c_4c_5)}}{4c_2c_4}\right\} \tag{3.21}$$

Then $\mathrm{v}_0 > \sqrt{\frac{2c_3+c_5}{2c_2}}$ and for all $\mathrm{v} \geq \mathrm{v}_0$

$$\frac{1}{c_2\mathrm{v}^2 - c_3 - \frac{c_5}{2}} \leq \frac{2}{c_5} \quad \text{and} \quad \frac{c_1c_5(\mathrm{v}+1)}{2c_2\mathrm{v}^2 - 2c_3 - c_5} \leq c_4.$$

This leads to the conclusion that whenever

$$\|\nabla\mathcal{A}^H(v,\eta)\| < \frac{c_5}{2} \quad \text{and} \quad \|v\|_{L^2} \geq \mathrm{v}_0,$$

then

$$|\eta| \leq \frac{2}{c_5}|\mathcal{A}^H(v,\eta)| + c_4.$$

Step 2:
Let us now take $(v,\eta) \in C^\infty(S^1,\mathbb{R}^{2n}) \times \mathbb{R}$, such that

$$\|\nabla\mathcal{A}^H(v,\eta)\| < \varepsilon, \quad \text{and} \quad \|v\|_{L^2} \leq \mathrm{v}_0,$$

The fact that

$$\varepsilon < \frac{\nu}{2}\min\left\{1, \frac{1}{M\mathrm{v}_0 + h_1}\right\},$$

by Lemma 3.6 implies that

$$v(t) \in H^{-1}(-\nu,\nu) \qquad \forall\, t \in [0,1].$$

Therefore, we can assume that the Liouville vector field $X^\ddagger$ of the Hamiltonian H is defined along the whole loop v and using Property (H3) we get the following relation

$$\mathcal{A}^H(v,\eta) - {\mathcal{A}'}^H_{(v,\eta)}(X^\ddagger,\eta) = \eta\int dH(X^\ddagger),$$

$$|\mathcal{A}^H(v,\eta)| + \|\nabla\mathcal{A}^H(v,\eta)\|(c_4(\|v\|_{L^2}^2+1) + |\eta|) \geq c_5|\eta|,$$

$$|\mathcal{A}^H(v,\eta)| + \varepsilon c_4(\mathrm{v}_0^2+1) \geq (c_5-\varepsilon)|\eta|.$$

Since by assumption $\varepsilon < \frac{c_5}{2}$, we obtain the claimed relation namely

$$|\eta| \leq \frac{2}{c_5}|\mathcal{A}^H(v,\eta)| + c_4(\mathrm{v}_0^2+1).$$

□

Lemma 3.6. *Fix $H_0 \in C^\infty(\mathbb{R}^{2n})$ satisfying properties (H1), (H2) and (H3) and a compact set $K \subseteq \mathbb{R}^{2n}$, $K \neq \emptyset$. Let $\mathscr{O}(H_0) \subseteq C_0^\infty(K)$ be the open set as in Lemma 3.2 and fix a Hamiltonian $H \in H_0 + \mathscr{O}(H_0)$. Then for $(v, \eta) \in C^\infty(S^1, \mathbb{R}^{2n}) \times \mathbb{R}$, the condition*

$$\|\nabla \mathcal{A}^H(v,\eta)\| < \varepsilon \leq \frac{\mu}{2}$$

implies that there exists $t_0 \in [0,1]$ such that $v(t_0) \in H^{-1}(-\frac{\mu}{2}, \frac{\mu}{2}))$. Moreover, if we assume $\|v\|_{L^2} \leq \mathrm{v}$ then

$$v(t) \in H^{-1}(-\mu, \mu)) \quad \forall\, t \in [0,1],$$

whenever

$$\varepsilon \leq \frac{\mu}{2} \min\Big\{1, \frac{1}{h_1 + M\mathrm{v}}\Big\}.$$

Proof. Assume that $v \in C^\infty(S^1, \mathbb{R}^{2n})$ has the property that $|H(v(t))| \geq \frac{\mu}{2}$ for all $t \in [0,1]$. Then

$$\|\nabla \mathcal{A}^H(v,\eta)\| \geq \Big| \int_0^1 H(v(t))dt \Big| \geq \frac{\mu}{2}.$$

In particular, if we take $|\nabla \mathcal{A}^H(v,\eta)\| < \varepsilon \leq \frac{\mu}{2}$ we get a contradiction, unless there exists a $t_0 \in [0,1]$ such that

$$v(t_0) \in H^{-1}\Big(-\frac{\mu}{2}, \frac{\mu}{2}\Big).$$

Let us now prove the second part of the lemma. By assumption the assertions of Lemma 3.2. hold true. In particular, if $\|v\|_{L^2} \leq \mathrm{v}$, then we have that the gradient of H is uniformly bounded

$$\|\nabla H(v)\|_{L^2} \leq h_1 + M\|v\|_{L^2} \leq h_1 + M\mathrm{v}.$$

Therefore, for every $t \in [0,1]$

$$\begin{aligned}
(h_1 + M\mathrm{v})\ \|\nabla \mathcal{A}^H(v,\eta)\| &\geq \|\nabla H(v)\|_{L^2} \|\partial_t v - \eta X^H(v)\|_{L^2} \\
&\geq \int_0^1 \|\nabla H(v(\tau))\|\ \|\partial_t v(\tau) - \eta X^H(v(\tau))\| d\tau \\
&\geq \int_{t_0}^t \|\nabla H(v(\tau))\|\ \|\partial_t v(\tau) - \eta X^H(v(\tau))\| d\tau \\
&\geq \int_{t_0}^t |\langle \nabla H(v(\tau)), \partial_t v(\tau) - \eta X^H(v(\tau))\rangle| d\tau \\
&= \int_{t_0}^t |dH(\partial_t v(\tau))| d\tau \\
&\geq \Big| \int_{t_0}^t dH(\partial_t v(\tau)) d\tau \Big| \\
&= |H(v(t)) - H(v(t_0))| \\
&\geq \big| |H(v(t))| - |H(v(t_0))| \big|.
\end{aligned}$$

So now if we take
$$\varepsilon \leq \frac{\mu}{2} \min\Big\{1, \frac{1}{M\mathrm{v}+h_1}\Big\},$$
then for $(v,\eta) \in C^\infty(S^1,\mathbb{R}^{2n}) \times \mathbb{R}$, such that $\|\nabla \mathcal{A}^H(v,\eta)\| < \varepsilon$ and $\|v\|_{L^2} \leq \mathrm{v}$ we will have that
$$\frac{\mu}{2} \geq \varepsilon(M\mathrm{v}+h_1) > \big||H(v(t))| - |H(v(t_0))|\big|,$$
$$\text{and} \qquad v(t_0) \in H^{-1}\Big(-\frac{\mu}{2}, \frac{\mu}{2}\Big),$$
so in fact
$$v(t) \in H^{-1}(-\mu,\mu) \qquad \forall\, t \in [0,1].$$

□

3.5 Set of infinitesimal action derivation

Let us introduce the set of infinitesimal action derivation $\mathcal{B}^J(a,b,\varepsilon)$.

Definition 3.7. *Let* $\Gamma = \{H_s, J_s\}_{s\in\mathbb{R}}$ *be a smooth homotopy of Hamiltonians and almost complex structures, constant outside of* $[0,1]$ *and let* $J_s \in \mathscr{J}(\mathbb{R}^{2n},\omega_0)$ *for all* $s \in \mathbb{R}$.

Then for every triple of constants $0 < \varepsilon, \mathfrak{a}, \mathfrak{y} < \infty$ *we define the* ***set of infinitesimal action derivation*** *of* Γ *as*

$$\mathcal{B}^J(\mathfrak{a},\mathfrak{y},\varepsilon) := \Big\{(v,\eta) \in C^\infty(S^1,\mathbb{R}^{2n}) \times \mathbb{R} \;\Big|\; |\eta| \leq \mathfrak{y}, \\ \exists\, s \in [0,1]\ |\mathcal{A}^{H_s}(v,\eta)| \leq \mathfrak{a},\ \&\ \|\nabla^{J_s}\mathcal{A}^{H_s}(v,\eta)\|_{L^2\times\mathbb{R}} \leq \varepsilon\Big\}.$$

By Proposition 3.4 for a fixed pair $a, b \in \mathbb{R}$ there exist constants $0 < \mathfrak{a}, \mathfrak{y} < \infty$, such that for any pair of connected components
$$\Lambda_0 \subseteq \mathscr{C}(\mathcal{A}^{H_0}, N) \cap (\mathcal{A}^{H_0})^{-1}([a,\infty)),$$
$$\Lambda_1 \subseteq Crit(\mathcal{A}^{H_1}) \cap (\mathcal{A}^{H_1})^{-1}((-\infty,b]),$$
the corresponding Floer trajectories in $\mathscr{M}^\Gamma(\Lambda_0,\Lambda_1)$ have their the action and η parameter uniformly bounded, i.e.:
$$\sup_{s\in\mathbb{R}} |\mathcal{A}^{H_s}(u(s))| \leq \mathfrak{a}, \qquad \sup_{s\in\mathbb{R}} |\eta(s)| \leq \mathfrak{y}.$$
This implies that every such Floer trajectory starts and ends in $\mathcal{B}^J(\mathfrak{a},\mathfrak{y},\varepsilon)$ for all $\varepsilon > 0$. In other words, if $u \in \mathscr{M}^\Gamma(\Lambda_0,\Lambda_1)$, then
$$\forall\, \varepsilon > 0,\ \exists\, S_\varepsilon > 0,\ \forall\, |s| > S_\varepsilon \qquad u(s) \in \mathcal{B}^J(\mathfrak{a},\mathfrak{y},\varepsilon).$$

Observe that for each pair $\mathfrak{a}, \mathfrak{y} > 0$ a subset of critical points of $\mathcal{A}^{H_s}$ is contained in $\mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon)$, namely for all $s \in [0,1]$

$$\left\{(v,\eta) \in Crit(\mathcal{A}^{H_s}) \;\middle|\; |\eta| \leq \mathfrak{y} \;\&\; |\mathcal{A}^{H_s}(v,\eta)| \leq \mathfrak{a}\right\} \subseteq \mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon)$$

for all $\varepsilon > 0$.

In particular, for all $\mathfrak{a}, \mathfrak{y} > 0$ and every $s \in [0,1]$ the whole 0-level-set of H_s is contained in $\mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon)$,

$$\forall\ s \in [0,1] \qquad H_s^{-1}(0) \times \{0\} \subseteq \mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon).$$

Therefore, for any triple $\mathfrak{a}, \mathfrak{y}, \varepsilon > 0$, whenever the $H_s^{-1}(0)$ are non-compact, then $\mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon)$ is certainly not bounded.

An important step in finding the bounds on Floer trajectories is to localize $\mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon)$. The time that a Floer trajectory spends outside of $\mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon)$ is bounded by the energy, as discussed in Lemma 3.17. Unfortunately, the time that a Floer trajectory spends inside of $\mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon)$ cannot be bounded this way, so in principle the trajectories could escape to infinity inside $\mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon)$, since it is not bounded.

Denote by $\Sigma \subseteq C^\infty(S^1, \mathbb{R}^{2n})$ the set of constant loops on $H_0^{-1}(0)$ and by $\mathcal{U}^1_\delta$ the $W^{1,2} \times \mathbb{R}$ neighborhood of $\Sigma \times \{0\}$ defined by

$$\mathcal{U}^1_\delta := \{(v,\eta) \in C^\infty(S^1, \mathbb{R}^{2n}) \times \mathbb{R} \mid dist_{W^{1,2}\times\mathbb{R}}((v,\eta), \Sigma \times \{0\}) < \delta\}, \tag{3.22}$$

In the following Proposition we will prove that for every $\mathfrak{a}, \mathfrak{y}, \delta > 0$ there exists $\varepsilon > 0$ and a set $K^1_\delta \subseteq C^\infty(S^1, \mathbb{R}^{2n})$, bounded in $W^{1,2} \times \mathbb{R}$ norm, such that

$$\mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon) \subseteq K_\delta \cup \mathcal{U}^1_\delta,$$

provided $\|J\|_\infty < +\infty$. In other words if $(v,\eta) \in \mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon)$, then either v is bounded in L^∞ or $dist_{W^{1,2}}(v, \Sigma) < \delta$.

Proposition 3.8. *Fix $H_0 \in C^\infty(\mathbb{R}^{2n})$ satisfying properties (H1), (H2) and (H3) and a compact set $K \subseteq \mathbb{R}^{2n}$, $K \neq \emptyset$ and let $\mathscr{O}(H_0) \subseteq C_0^\infty(K)$ be the open set as in Lemma 3.2. Let $\Gamma = \{H_s, J_s\}_{s\in\mathbb{R}}$ be a smooth homotopy of Hamiltonians and almost complex structures, constant outside $[0,1]$, such that*

$$\forall\ s \in \mathbb{R} \qquad (H_s, J_s) \in (H_0 + \mathscr{O}(H_0)) \times \mathscr{J}(\mathbb{R}^{2n}, \omega_0).$$

Then for all triples $\mathfrak{a}, \mathfrak{y} < \infty$, $\delta > 0$ and $r > \sup_{x\in K}\|x\|$, there exist $\varepsilon_2(\delta, \|J\|_\infty) > 0$ and $\mathrm{v}_2(\delta, r) > 0$, such that for all $\varepsilon < \varepsilon_2(\delta)$ and $(v,\eta) \in \mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon)$ one of the following holds:

1. *If $\|v\|_{L^2} \geq \mathrm{v}_2$ then*

$$\|v(t)\| \geq r \quad \forall\ t \in S^1,$$
$$dist_{W^{1,2}\times\mathbb{R}}((v,\eta), \Sigma \times \{0\}) < \delta,$$

where Σ is the set of constant loops on $H_0^{-1}(0)$.

2. If $\|v\|_{L^2} \leq \mathrm{v}_2$, then in fact (v,η) is uniformly bounded in $W^{1,2} \times \mathbb{R}$.

Proof. First observe that since $H_s \in H_0 + \mathscr{O}(H_0)$ for every $s \in \mathbb{R}$, therefore by Lemma 3.2 we can choose the parameters from Definition 3.1 uniformly for the whole set $H_0 + \mathscr{O}(H_0)$.

Let us fix $\delta > 0$ and take $\varepsilon_0 > 0$ as in Lemma 3.5, $\mathrm{v}_1(\frac{\delta}{4}), \varepsilon_1(\frac{\delta}{4}) > 0$ as in Lemma 3.10 and $\mu(\frac{\delta}{4}, H_0) > 0$ as in Lemma 4.13. Then we claim that the statement of the Proposition holds for $\varepsilon_2(\delta), \mathrm{v}_2(\delta, r) > 0$ defined as below

$$\begin{aligned}\mathrm{v}_2(\delta, r) &:= \max\Big\{\mathrm{v}_1\Big(\frac{\delta}{4}\Big), r + \frac{1}{4}\delta\Big\},\\ \varepsilon_2(\delta, \|J\|_\infty) &:= \frac{\varepsilon_2(\delta)}{\max\{1, \|J\|_\infty\}},\\ \varepsilon_2(\delta) &:= \min\Big\{\varepsilon_0, \varepsilon_1\Big(\frac{\delta}{4}\Big), \mu\Big(\frac{\delta}{4}, H_0\Big)\Big\}.\end{aligned}$$

Note that for every $J \in \mathscr{J}(\mathbb{R}^{2n}, \omega_0)$ we have

$$\min\{1, \|J\|_\infty^{-1}\} \|\nabla \mathcal{A}^H(v,\eta)\|_{L^2\times\mathbb{R}} \leq \|\nabla^J \mathcal{A}^H(v,\eta)\|_{g_J}$$

which implies

$$\mathcal{B}^J\Big(\mathfrak{a}, \mathfrak{y}, \varepsilon_2(\delta, \|J\|_\infty)\Big) \subseteq \mathcal{B}^{J_0}(\mathfrak{a}, \mathfrak{y}, \varepsilon_2(\delta)).$$

Therefore, without loss of generality, one can assume the J structure to be constant and equal everywhere to J_0.

Part 1:

Fix $\varepsilon < \varepsilon_2(\delta)$ and take $(v,\eta) \in \mathcal{B}^{J_0}(\mathfrak{a}, \mathfrak{y}, \varepsilon)$, such that $\|v\|_{L^2} \geq \mathrm{v}_2(r, \delta)$.

For every $t_0 \in S^1$ one has

$$\begin{aligned}\|v - v(t_0)\|_{L^2} &\leq \Big(\int_0^1 \|v(t) - v(t_0)\|^2 dt\Big)^{\frac{1}{2}}\\ &= \Big(\int_0^1 \Big\|\int_{t_0}^t \partial_t v(\tau) d\tau\Big\|^2 dt\Big)^{\frac{1}{2}}\\ &\leq \|\partial_t v\|_{L^2}\end{aligned}$$

Since $\varepsilon < \varepsilon_2(\delta) \leq \varepsilon_1\big(\frac{\delta}{4}\big)$ and $\|v\|_{L^2} \geq \mathrm{v}_2(r,\delta) \geq \mathrm{v}_1\big(\frac{\delta}{4}\big)$ so by Lemma 3.10.

$$\frac{\delta}{4} \geq \|\partial_t v\|_{L^2} \geq \|v - v(t_0)\|_{L^2}.$$

Taking moreover into account that $\|v\|_{L^2} \geq \mathrm{v}_2(r,\delta) \geq r + \frac{\delta}{4}$ we obtain the bound

$$\|v(t_0)\| \geq \|v\|_{L^2} - \|v - v(t_0)\|_{L^2} \geq r + \frac{\delta}{4} - \frac{\delta}{4} = r.$$

Since we chose $r > \sup_{x\in K}\|x\|$, it follows that

$$(v,\eta) \in \mathcal{B}^{J_0}(\mathfrak{a}, \mathfrak{y}, \varepsilon) \quad \& \quad \|v\|_{L^2} \geq \mathrm{v}_2(\delta) \quad \Longrightarrow \quad v(t) \notin K \quad \forall\, t \in S^1.$$

$v(t) \notin K$ for all $t \in S^1$, hence for all $s \in [0,1]$, $H_s(v(t)) = H_0(v(t))$. Together with Lemma 3.6, this gives us that there exists $t_0 \in [0,1]$ such that

$$v(t_0) \in H_0^{-1}(-\varepsilon_2(\delta), \varepsilon_2(\delta))$$

Since $\varepsilon < \varepsilon_2(\delta) \leq \mu\Big(\frac{\delta}{4}, H_0\Big)$, then by Lemma 4.13 there exists $t_0 \in [0,1]$ such that

$$dist(v(t_0), H_s^{-1}(0)) < \frac{\delta}{4}.$$

Analogically, one can we use Lemma 3.10. to estimate

$$\begin{aligned} dist_{L^2}(v, \Sigma) &\leq dist(v(t_0), H^{-1}(0)) + \|v(t_0) - v\|_{L^2} \\ &\leq dist(v(t_0), H^{-1}(0)) + \|\partial_t v\|_{L^2} \\ &\leq \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2}. \end{aligned}$$

By Lemma 3.10. we have also $|\eta| \leq \frac{\delta}{4}$. Therefore, for $(v, \eta) \in \mathcal{B}^{J_0}(\mathfrak{a}, \mathfrak{y}, \varepsilon_2(\delta))$ and $\|v\|_{L^2} \geq \mathrm{v}_2(r, \delta)$ one has

$$dist_{W^{1,2}\times\mathbb{R}}((v,\eta), \Sigma \times \{0\}) \leq dist_{L^2}(v, \Sigma) + \|\partial_t v\|_{L^2} + |\eta| \leq \delta.$$

Part 2:

Since $H_s \in H_0 + \mathscr{O}(H_0)$ for all $s \in [0,1]$, by Lemma 3.2 there exist $h_1, M \in \mathbb{R}$ such that

$$\|\nabla H_s(x)\| \leq h_1 + M\|x\| \qquad \forall\, x \in \mathbb{R}^{2n}.$$

Take $(v, \eta) \in \mathcal{B}^{J_0}(\mathfrak{a}, \mathfrak{y}, \varepsilon_2(\delta))$ and $\|v\|_{L^2} \leq \mathrm{v}_2(r, \delta)$. By assumption $|\eta| \leq \mathfrak{y}$ so we have the following estimate:

$$\begin{aligned} \|\partial_t v\|_{L^2} &\leq \|\partial_t v - \eta X^{H_s}\|_{L^2} + |\eta| \|\nabla H_s(v)\|_{L^2} \\ &\leq \varepsilon_2(\delta) + \mathfrak{y}(h_1 + M\|v\|_{L^2}) \\ &\leq \varepsilon_2(\delta) + \mathfrak{y}(h_1 + M\mathrm{v}_2(r, \delta)), \end{aligned}$$

It follows that $\|(v,\eta)\|_{W^{1,2}\times\mathbb{R}}$ is uniformly bounded as claimed. Moreover, the bounds are uniform for the whole set $H_0 + \mathscr{O}(H_0)$ and depend continuously on $\mathfrak{y}$. $\square$

Observe that the bounds and constants in the proposition above do not depend on the choice of the homotopy Γ, but on the fact that $H_s \in H_0 + \mathscr{O}(H_0)$, which allows to choose uniform parameters from Definition 3.1. Only $\varepsilon_2(\delta, \|J\|_\infty)$ depends on $\|J\|_\infty$ explicitly and the bounds on $\|\partial_s v\|_{L^2}$ in Part 2 depend on $\mathfrak{y}$, which through Proposition 3.4 depends on $\|J\|_\infty$, but both of these quantities depend on $\|J\|_\infty$ continuously.

Now, in the following lemma, we show that every homotopy Γ defined as in Theorem 7 satisfies Novikov finiteness condition. As a result, for every pair $a, b \in \mathbb{R}$ there exists a bounded subset of $\mathbb{R}^{2n+1}$ which contains the beginnings of all Floer trajectories in $\mathscr{M}^\Gamma(\Lambda_0, \Lambda_1)$ provided

$$\Lambda_0 \subseteq \mathscr{C}(\mathcal{A}^{H_0}, N) \cap (\mathcal{A}^{H_0})^{-1}([a, \infty)) \quad \text{and} \quad \Lambda_1 \subseteq Crit(\mathcal{A}^{H_1}) \cap (\mathcal{A}^{H_1})^{-1}((-\infty, b]).$$

Lemma 3.9. *Let $\Gamma = \{H_s, J_s\}_{s\in\mathbb{R}}$ be a smooth homotopy of Hamiltonians and almost complex structures defined as in Theorem 7. Then for every pair $a, b \in \mathbb{R}$ and a compact subset $N_0 \subseteq N \subseteq H_0^{-1}(0)$ there exists a compact subset of $\mathbb{R}^{2n+1}$, which contains all the connected components*

$$\Lambda_0 \subseteq \mathscr{C}(\mathcal{A}^{H_0}, N) \cap (\mathcal{A}^{H_0})^{-1}([a, \infty))$$

for which there exists a connected component

$$\Lambda_1 \subseteq Crit(\mathcal{A}^{H_1}) \cap (\mathcal{A}^{H_1})^{-1}((-\infty, b])$$

such that

$$\mathscr{M}^{\Gamma}(\Lambda_0, \Lambda_1) \neq \emptyset.$$

Proof. Since $\Gamma = \{H_s, J_s\}_{s\in\mathbb{R}}$ is a homotopy defined as in Theorem 7, we can apply Lemma 3.5 to prove the linearity condition between the action and the η as stated in (3.6). Having established (3.6), we can directly apply Corollary 3.8 from [12], which gives us a relation between $\mathcal{A}^{H^0}(\Lambda^0)$ and $\mathcal{A}^{H^1}(\Lambda^1)$, namely if $\mathscr{M}^{\Gamma}(\Lambda_0, \Lambda_1) \neq \emptyset$, then

$$a \leq \max\{2b, 1\} \qquad \text{and} \qquad b \geq \min\{2a, -1\}.$$

In particular, Γ satisfies the Novikov finiteness condition:

$$\begin{aligned} A(\Gamma, a, b) &\geq \min\{2a, -1\} \\ B(\Gamma, a, b) &\leq \max\{2b, 1\}, \end{aligned}$$

where $A(\Gamma, a, b)$ and $B(\Gamma, a, b)$ defined in (2.23) and (2.24) respectively.

By assumption H_0 and H_1 satisfy (PO), hence all the connected components

$$\Lambda_0 \subseteq Crit(\mathcal{A}^{H_0}) \cap (\mathcal{A}^{H_0})^{-1}([a, \max\{2b, 1\}] \setminus \{0\})$$

have the property that

$$N \bigcup_{(v,\eta)\in\Lambda_0} v(S^1)$$

is a bounded subset of $H_0^{-1}(0)$. Define

$$\mathrm{v}_3 := \sup\{\|v\|_{L^\infty} \mid (v, \eta) \in \mathscr{C}(\mathcal{A}^{H_0}, N) \cap (\mathcal{A}^{H_0})^{-1}([a, \max\{2b, 1\}] \setminus \{0\})\}. \tag{3.23}$$

Then combining with the boundedness of η by $\mathfrak{y}$ established in Proposition 3.4 and Property (H2), we obtain the uniform boundedness in the $W^{1,2} \times \mathbb{R}$ norm namely for all (v, η) in

$$\mathscr{C}(\mathcal{A}^{H_0}, N) \cap (\mathcal{A}^{H_0})^{-1}([a, \max\{2b, 1\}])$$

one has

$$\begin{aligned} |\eta| &\leq \mathfrak{y} \\ \|v\|_{L^2} &\leq \mathrm{v}_3 \\ \|\partial_t v\|_{L^2} &\leq \mathfrak{y}(h_1 + M\mathrm{v}_3). \end{aligned}$$

□

As a conclusion, for every $\mathfrak{a}, \mathfrak{y}, \delta > 0$ if we define a subset of $C^\infty(S^1, \mathbb{R}^{2n})$ bounded in $W^{1,2} \times \mathbb{R}$ norm in the following way:

$$\mathcal{K}^1_\delta := \left\{ (v,\eta) \in C^\infty(S^1, \mathbb{R}^{2n}) \times \mathbb{R} \;\middle|\; \begin{array}{c} |\eta| \leq \mathfrak{y} \\ \|v\|_{L^2} \leq \max\{\mathrm{v}_2, \mathrm{v}_3\} \\ \|\partial_t v\|_{L^2} \leq \varepsilon_2 + \mathfrak{y}(h_1 + M \max\{\mathrm{v}_2, \mathrm{v}_3\}) \end{array} \right\} \tag{3.24}$$

where $\varepsilon_2(\delta, \|J\|_\infty) > 0$ and $\mathrm{v}_2(\delta, \max_{K \cup \mathcal{V}} \|x\|) < +\infty$ are defined as in Proposition 3.8 and v_3 is as in (3.23). As a result for every $\varepsilon < \varepsilon_2(\delta, \|J\|_\infty)$ one has

$$\begin{aligned} &\mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon) \subseteq K^1_\delta \cup \mathcal{U}^1_\delta, \\ &\mathscr{C}(\mathcal{A}^{H_0}, N) \cap (\mathcal{A}^{H_0})^{-1}([a, \max\{2b, 1\}] \setminus \{0\}) \subseteq K^1_\delta. \end{aligned}$$

Now we will present the proof of one of the lemmas used in Proposition 3.8. The proof of the other lemma used, namely Lemma 4.13 can be found in Chapter 4, since it relates more to the geometrical properties of the tentacular Hamiltonians.

Lemma 3.10. *Let $\Gamma = \{H_s, J_s\}_{s \in \mathbb{R}}$ be a smooth homotopy of Hamiltonians and almost complex structures as in Proposition 3.8. Fix $\mathfrak{a}, \mathfrak{y} > 0$. Then for every $\delta > 0$, there exist $\varepsilon_1(\delta), \mathrm{v}_1(\delta) > 0$, depending only on $\mathfrak{a}$ and δ, such that whenever $(v, \eta) \in \mathcal{B}^{J_0}(\mathfrak{a}, \mathfrak{y}, \varepsilon_1(\delta))$ and $\|v\|_{L^2} > \mathrm{v}_1(\delta)$, the derivative $\partial_t v$ and η are both bounded by δ*

$$|\eta| \leq \delta, \qquad \|\partial_t v\| \leq \delta.$$

Proof. By assumption

$$\forall\ s \in \mathbb{R} \qquad H_s \in H_0 + \mathscr{O}(H_0),$$

therefore by Lemma 3.2 the associated parameters can be chosen uniformly for the whole family.

Recall from (2.39) that for every Liouville vector field Y and Hamiltonian H we have

$$\mathcal{A}^H(v, \eta) - \mathcal{A}'^H_{(v,\eta)}(Y, \eta) = \eta \int dH(Y).$$

Let $X^\dagger$ be the Liouville vector field given by Property (H1) and Lemma 3.2 for all $H \in H_0 + \mathscr{O}(H_0)$. If we now apply $X^\dagger$ and Property (H1) to the above identity, we obtain

$$|\mathcal{A}^{H_s}(v,\eta)| + \|\nabla \mathcal{A}^{H_s}(v,\eta)\|(c_1(\|v\|_{L^2} + 1) + |\eta|) \geq |\eta|(c_2\|v\|^2_{L^2} - c_3).$$

Therefore for $(v, \eta) \in \mathcal{B}^{J_0}(\mathfrak{a}, \mathfrak{y}, \varepsilon)$ with $\|v\|_{L^2}$ large enough, that is

$$\|v\|_{L^2} > \sqrt{\frac{\varepsilon + c_3}{c_2}},$$

the following inequality holds

$$\frac{\mathfrak{a} + c_1 \varepsilon(\|v\|_{L^2} + 1)}{c_2\|v\|^2_{L^2} - c_3 - \varepsilon} \geq |\eta|.$$

Moreover from Property (H2) we have that

$$\|\nabla H_s(x)\| \leq h_1 + M\|x\| \quad \forall\, x \in \mathbb{R}^{2n}.$$

Therefore for $(v,\eta) \in \mathcal{B}^{J_0}(\mathfrak{a},\mathfrak{y},\varepsilon)$ one also has

$$\begin{aligned}\|\partial_t v\|_{L^2} &\leq \|\partial_t v - \eta X^{H_s}(v)\|_{L^2} + |\eta|\|\nabla H_s(v)\|_{L^2}\\ &\leq \varepsilon + \frac{\mathfrak{a} + \varepsilon(c_1\|v\|_{L^2}+1)}{c_2\|v\|_{L^2}^2 - c_3 - \varepsilon}(h_1 + M\|v\|_{L^2}).\end{aligned}$$

Note that this bound does not depend on s any more. If we define

$$\varepsilon_1(\delta) := \frac{\delta}{2}\min\{\frac{c_2}{2Mc_1}, 1\},$$

then we have

$$\begin{aligned}\|\partial_t v\|_{L^2} &\leq \varepsilon_1(\delta) + \frac{\mathfrak{a} + \varepsilon_1(\delta)c_1(\|v\|_{L^2}+1)}{c_2\|v\|_{L^2}^2 - c_3 - \varepsilon_1(\delta)}(h_1 + M\|v\|_{L^2})\\ &\leq \frac{\delta}{2} + \frac{\mathfrak{a} + \frac{\delta c_2}{4M}(\|v\|_{L^2}+1)}{c_2\|v\|_{L^2}^2 - c_3 - \delta\frac{c_2}{4Mc_1}}(h_1 + M\|v\|_{L^2}).\end{aligned}$$

If we analyze the right-hand side, we see that as $\|v\|_{L^2} \to \infty$,

$$\lim_{\|v\|_{L^2}\to\infty}\Big(\frac{\delta}{2} + \frac{\mathfrak{a} + \frac{\delta c_2}{4M}(\|v\|_{L^2}+1)}{c_2\|v\|_{L^2}^2 - c_3 - \delta\frac{c_2}{4Mc_1}}(h_1 + M\|v\|_{L^2})\Big) = \frac{3}{4}\delta.$$

Moreover,

$$\frac{\mathfrak{a} + \frac{\delta c_2}{4M}(\|v\|_{L^2}+1)}{c_2\|v\|_{L^2}^2 - c_3 - \delta\frac{c_2}{4Mc_1}} \geq |\eta|,$$

$$\text{and} \qquad \lim_{\|v\|_{L^2}\to\infty}\frac{\mathfrak{a} + \frac{\delta c_2}{4M}(\|v\|_{L^2}+1)}{c_2\|v\|_{L^2}^2 - c_3 - \delta\frac{c_2}{4Mc_1}} = 0.$$

Therefore there exists v_1 depending only on δ, such that if $\|v\|_{L^2} \geq v_1(\delta)$ then the following inequalities both hold:

$$\begin{aligned}|\eta| &\leq \frac{\mathfrak{a} + \frac{\delta c_2}{4M}(\|v\|_{L^2}+1)}{c_2\|v\|_{L^2}^2 - c_3 - \delta\frac{c_2}{4Mc_1}} \leq \delta,\\ \|v\|_{L^2} &\leq \frac{\delta}{2} + \frac{\mathfrak{a} + \frac{\delta c_2}{4M}(\|v\|_{L^2}+1)}{c_2\|v\|_{L^2}^2 - c_3 - \delta\frac{c_2}{4Mc_1}}(h_1 + M\|v\|_{L^2}) \leq \delta.\end{aligned}$$

In other words, for every $\delta > 0$ there exists $v_1(\delta) > 0$, such that whenever $(v,\eta) \in \mathcal{B}^{J_0}(\mathfrak{a},\mathfrak{y},\varepsilon_1(\delta))$ and $\|v\|_{L^2} \geq v_1(\delta)$, then

$$|\eta| \leq \delta \qquad \text{and} \qquad \|\partial_t v\| \leq \delta.$$

$\square$

3.6 Floer trajectories near the critical set

Having localized the set $\mathcal{B}^J(\mathfrak{a},\mathfrak{y},\varepsilon)$ in the previous chapter, we will now investigate the behavior of the Floer trajectories within $\mathcal{B}^J(\mathfrak{a},\mathfrak{y},\varepsilon)$. From Proposition 3.8, we know that for every $\mathfrak{a},\mathfrak{y},\delta>0$ there exists $\varepsilon>0$ and a set $K_\delta\subseteq C^\infty(S^1,\mathbb{R}^{2n})\times\mathbb{R}$, bounded in $W^{1,2}\times\mathbb{R}$ norm, such that

$$\mathcal{B}^J(\mathfrak{a},\mathfrak{y},\varepsilon)\subseteq K_\delta\cup\mathcal{U}^1_\delta.$$

Since $\mathcal{U}^1_\delta$ (defined in (3.22)) is non-compact, we will have to find a way to ensure that the Floer trajectories don't escape to ∞ within this set.

Let us now introduce a set defined analogically to $\mathcal{U}^1_\delta$, namely

$$\mathcal{U}^0_\delta:=\{x\in C^\infty(S^1,\mathbb{R}^{2n})\times\mathbb{R}\mid dist_{L^2\times\mathbb{R}}((v,\eta),\Sigma\times\{0\})<\delta\}. \tag{3.25}$$

In Proposition 3.16 we show that due to the Morse-Bott assumption on the action functional, there exists a $\delta>0$ such that the Floer trajectories in $\mathcal{U}^0_\delta$ cannot escape along Σ. Naturally, for every $\delta>0$, the following inclusion holds

$$\mathcal{B}^J(\mathfrak{a},\mathfrak{y},\varepsilon)\subseteq K_\delta\cup\mathcal{U}^1_\delta\subseteq K_\delta\cup\mathcal{U}^0_\delta,$$

thus ensuring that the Floer trajectories don't escape within $\mathcal{B}^J(\mathfrak{a},\mathfrak{y},\varepsilon)$. In fact, as we will prove in this section, due to the Morse-Bott property of the action functional $\mathcal{A}^H$, the tangential component of the Floer trajectories along the critical submanifold can be estimated by the energy growth.

To make this assertion more precise and to prove it, we will view $\mathcal{U}^0_\delta$ as a tubular neighborhood of $\Sigma\times\{0\}$ in $C^\infty(S^1,\mathbb{R}^{2n})\times\mathbb{R}$ with respect to the metric induced by J_0. Next, we will define a projection P of $\mathcal{U}^0_\delta$ onto $\Sigma\times\{0\}$ and then analyze the image of the Floer trajectories under this projection. For this analysis we will need the Taylor expansion of $\nabla^{J_0}\mathcal{A}^H$ in this tubular neighborhood with respect to the points on $\Sigma\times\{0\}$, which is the subject of the following subsections.

However, to simplify our computations we would like the Hamiltonian to be constant and the almost complex to be equal J_0. Therefore, we restrict our to the analysis of the Floer trajectories outside the interval $[0,1]$. Whenever $s\notin(0,1)$ then (H_s,J_s) is either $(H_0,\{J_{0,t}(\cdot,\eta)\}_{(t,\eta)\in S^1\times\mathbb{R}})$ or $(H_1,\{J_{1,t}(\cdot,\eta)\}_{(t,\eta)\in S^1\times\mathbb{R}})$. Moreover, recall that we chose our almost complex structures for all $s\in\mathbb{R}$

$$J_s=\{J_{s,t}(\cdot,\eta)\}_{S^1\times\mathbb{R}}\in\mathcal{J}\Big(\mathbb{R}^{2n},\omega_0,\mathcal{V}\times\big((-\infty,\mathfrak{y})\cup(\mathfrak{y},\infty)\big)\Big)$$

to be constant and equal to J_0 outside the open set $\mathcal{V}\times((-\infty,\mathfrak{y})\cup(\mathfrak{y},\infty))$. Therefore for any $\delta\in(0,\mathfrak{y})$ the almost complex structures J_s are constant and equal J_0 in $\mathcal{U}^0_\delta$. In other words, for all

$$J_{s,t}(v(t),\eta)=J_0\quad\forall\ (v,\eta)\in\mathcal{U}^0_\delta\quad\forall\ t\in S^1\qquad\forall\ s\in\mathbb{R}.$$

Therefore, throughout this section we assume $s\notin(0,1)$ and take $(H,J)=(H_0,J_0)$ or $(J,H)=(H_1,J_0)$, which will simplifies the setting and calculations significantly.

3.6.1 Tubular neighborhood and projection onto the critical set

The aim of this subsection is to construct the a tubular neighborhood of $\Sigma \times \{0\}$ in $C^\infty(S^1, \mathbb{R}^{2n}) \times \mathbb{R}$ along with projection P onto $\Sigma \times \{0\}$ and then analyze the properties of this projection.

For the submanifold

$$\Sigma \times \{0\} \subseteq C^\infty(S^1, \mathbb{R}^{2n}) \times \mathbb{R},$$

one can define the normal bundle $N(\Sigma \times \{0\})$ in the following way

$$\begin{aligned} N_{(v,0)}(\Sigma \times \{0\}) &:= T_{(v,0)}(C^\infty(S^1, \mathbb{R}^{2n}) \times \mathbb{R}) \big/ T_{(v,0)}(\Sigma \times \{0\}), \\ N(\Sigma \times \{0\}) &:= \{((v,0),(\xi,\sigma)) \mid v \in \Sigma,\ (\xi,\sigma) \in N_{(v,0)}(\Sigma \times \{0\})\}. \end{aligned}$$

Let us fix a metric g_{J_0} on $C^\infty(S^1, \mathbb{R}^{2n}) \times \mathbb{R}$ induced by the standard almost complex structure J_0

$$g_{J_0\ (v,\eta)}((\xi_1,\sigma_1),(\xi_2,\sigma_2)) := \int_0^1 \omega_0(\xi_1(t), J_0(v(t))\xi_2(t))dt + \sigma_1\sigma_2,$$

for $(\xi_1,\sigma_1),(\xi_2,\sigma_2) \in T_{(v,\eta)}C^\infty(S^1, \mathbb{R}^{2n}) \times \mathbb{R}$. In particular, one has:

$$\|(\xi,\sigma)\|_{g_{J_0}} = \|(\xi,\sigma)\|_{L^2 \times \mathbb{R}}.$$

Given g_{J_0} one can make the following identification:

$$\begin{aligned} N_{(v,0)}(\Sigma \times \{0\}) &\cong \left\{ (\xi,\sigma) \in C^\infty(S^1, v^*T\mathbb{R}^{2n}) \times \mathbb{R} \;\middle|\; \begin{array}{c} g_{J_0\ (v,\eta)}((\xi,\sigma),(\hat{\xi},\hat{\sigma})) = 0, \\ \forall\ (\hat{\xi},\hat{\sigma}) \in T_{(v,0)}(\Sigma \times \{0\}) \end{array} \right\} \\ &= \left\{ \xi \in C^\infty(S^1, v^*T\mathbb{R}^{2n}) \;\middle|\; \begin{array}{c} \int_0^1 \omega_0(\xi(t), J_0(v(t))\hat{\xi})dt = 0, \\ \forall\ \hat{\xi} \in T_v H^{-1}(0) \end{array} \right\} \times \mathbb{R} \\ &= \left\{ \xi \in C^\infty(S^1, v^*T\mathbb{R}^{2n}) \;\middle|\; \begin{array}{c} \langle \int_0^1 \xi(t)dt, \hat{\xi}\ \rangle = 0, \\ \forall\ \hat{\xi} \in T_v H^{-1}(0) \end{array} \right\} \times \mathbb{R}. \end{aligned}$$

Recall that the exponential map for the metric g_{J_0} is defined as follows

$$\begin{aligned} exp : T(C^\infty(S^1, \mathbb{R}^{2n}) \times \mathbb{R}) &\to C^\infty(S^1, \mathbb{R}^{2n}) \times \mathbb{R} \\ ((v(t),\eta),(\xi(t),\sigma)) &\mapsto (v(t)+\xi(t), \eta+\sigma). \end{aligned}$$

Additionally, due to the absence of curvature for g_{J_0} and the linear structure of $C^\infty(S^1, \mathbb{R}^{2n}) \times \mathbb{R}$, the parallel transport along geodesics can be viewed as an identity isomorphism under the natural identification of $C^\infty(S^1, v_1^*T\mathbb{R}^{2n}) \times \mathbb{R}$ with $C^\infty(S^1, v_2^*T\mathbb{R}^{2n}) \times \mathbb{R}$,

$$\begin{aligned} Pt^\gamma : C^\infty(S^1, v_1^*T\mathbb{R}^{2n}) \times \mathbb{R} &\to C^\infty(S^1, v_2^*T\mathbb{R}^{2n}) \times \mathbb{R}, \\ ((v_1(t),\xi(t)),\sigma) &\mapsto ((v_2(t),\xi(t)),\sigma). \end{aligned}$$

Let us now construct the tubular neighborhood of $\Sigma \times \{0\}$ in $C^\infty(S^1, \mathbb{R}^{2n}) \times \mathbb{R}$. For $\delta > 0$, define

$$N^\delta(\Sigma \times \{0\}) := \{((v,0),(\xi,\sigma)) \mid v \in \Sigma,\ (\xi,\sigma) \in N_{(v,0)}(\Sigma \times \{0\}),\ \|(\xi,\sigma)\|_{g_{J_0}} < \delta\}.$$

Restricting the exponential map to $N^\delta(\Sigma \times \{0\})$ we construct a tubular neighborhood $\mathscr{N}^\delta(\Sigma)$ of $\Sigma \times \{0\}$ in $C^\infty(S^1, \mathbb{R}^{2n}) \times \mathbb{R}$ that is if $\Phi : N^\delta(\Sigma \times \{0\}) \to C^\infty(S^1, \mathbb{R}^{2n}) \times \mathbb{R}$ denotes the restriction of the exponential map to the bundle $N^\delta(\Sigma \times \{0\})$, then we define the tubular neighborhood

$$\mathscr{N}^\delta(\Sigma) := \Phi(N^\delta(\Sigma \times \{0\})). \tag{3.26}$$

Locally Φ is a diffeomorphism. Whether there exists a $\delta > 0$, such that Φ is a global diffeomorphism, depends on the geometry of $H^{-1}(0)$. In Lemma 4.14 we show that such a $\delta > 0$ exists for H satisfying Properties (H1), (H2) and (H3).

In the following lemma we will relate the tubular neighborhood $\mathscr{N}^\delta(\Sigma)$ with the neighborhood $\mathcal{U}_\delta^0$ defined in (3.22).

Lemma 3.11. *Let $\delta > 0$ be small enough so that $\mathscr{N}^\delta(\Sigma)$ is well defined. Then*

$$\mathscr{N}^\delta(\Sigma) = \mathcal{U}_\delta^0.$$

Proof. Observe, that for every $(v,\eta) \in C^\infty(S^1, \mathbb{R}^{2n}) \times \mathbb{R}$ there exists $\bar{v} \in \Sigma$ such that

$$\Big\| \int_0^1 v(t)dt - \bar{v} \Big\| = \inf_{\hat{v} \in \Sigma} \Big\| \int_0^1 v(t)dt - \hat{v} \Big\|$$

and for this $\bar{v}$

$$\langle \int_0^1 v(t)dt - \bar{v}, \hat{\xi} \rangle = 0 \quad \forall\, \hat{\xi} \in T_{\hat{v}}\Sigma.$$

In other words for every $(v,\eta) \in C^\infty(S^1, \mathbb{R}^{2n}) \times \mathbb{R}$ there exists a $\bar{v} \in \Sigma$ such that

$$(v - \bar{v}, \eta) \in N_{(\bar{v},0)}(\Sigma \times \{0\})$$

On the other hand for $\xi \in C^\infty(S^1, \mathbb{R}^{2n})$ one has

$$\langle \int_0^1 \xi dt, \xi - \int_0^1 \xi dt \rangle_{L^2} = 0,$$
$$\|\xi\|_{L^2}^2 = \| \int_0^1 \xi dt \|^2 + \|\xi - \int_0^1 \xi dt\|_{L^2}^2,$$

therefore for $(v,\eta) \in C^\infty(S^1, \mathbb{R}^{2n}) \times \mathbb{R}$ one has

$$\begin{aligned} dist_{L^2 \times \mathbb{R}}((v,\eta), \Sigma \times \{0\}) &= \inf_{\hat{v} \in \Sigma} \Big(\|v - \hat{v}\|_{L^2}^2 + |\eta|^2 \Big)^{\frac{1}{2}} \\ &= \Big(\inf_{\hat{v} \in \Sigma} \Big\| \int_0^1 v(t)dt - \hat{v} \Big\|^2 + \Big\| v - \int_0^1 v(t)dt \Big\|_{L^2}^2 + |\eta|^2 \Big)^{\frac{1}{2}}. \end{aligned}$$

Combining the two results above we obtain that

$$
\begin{aligned}
(v,\eta)\in\mathcal{U}^0_\delta &\iff dist_{L^2\times\mathbb{R}}((v,\eta),\Sigma\times\{0\})\le\delta\\
&\iff \exists\,\bar v\in\Sigma\quad (dist_{L^2}(v,\bar v)^2+\eta^2)^{\frac12}=dist_{L^2\times\mathbb{R}}((v,\eta),\Sigma\times\{0\})\le\delta\\
&\iff \exists\,\bar v\in\Sigma\quad (v-\bar v,\eta)\in N_{(\bar v,0)}(\Sigma\times\{0\})\ \&\ \|(v-\bar v,\eta)\|_{L^2\times\mathbb{R}}\le\delta\\
&\iff \exists\,\bar v\in\Sigma\quad \text{such that}\quad ((\bar v,0),(v-\bar v,\eta))\in N^\delta(\Sigma\times\{0\})\\
&\iff (v,\eta)\in\mathscr{N}^\delta(\Sigma).
\end{aligned}
$$

□

With the tubular neighborhood $\mathscr{N}^\delta(\Sigma)$ defined in (3.26), one can associate a projection P defined uniquely in the following way:

$$
\begin{aligned}
&P:\mathscr{N}^\delta(\Sigma)\to\Sigma\times\{0\},\\
\text{such that}\qquad &P\circ\Phi((v,0),(\xi,\sigma))=(v,0),\qquad \forall\ ((v,0),(\xi,\sigma))\in N^\delta(\Sigma\times\{0\}).
\end{aligned}
$$

Let us now take a closer look at the projection P and prove some of its properties that will become useful later in the analysis of the projection of the Floer trajectory on $\Sigma\times\{0\}$ and in the proof of Proposition 3.16.

Lemma 3.12. *Let P be the projection associated with the tubular neighborhood $\mathscr{N}^\delta(\Sigma)$. Then for every $(v,\eta)\in\mathscr{N}^\delta(\Sigma)$*

$$
N_{P(v,\eta)}(\Sigma\times\{0\})\subseteq Ker(dP_{(v,\eta)}\circ Pt^\gamma_{P(v,\eta)}),
$$

where Pt^γ is the parallel transport between $P(v,\eta)$ and (v,η).

Proof. Take $(v,\eta)\in\mathscr{N}^\delta(\Sigma)$. By definition there exists $(\xi,\sigma)\in N^\delta_{P(v,\eta)}(\Sigma\times\{0\})$, such that $\Phi(P(v,\eta),(\xi,\sigma))=(v,\eta)$. For every $(\tilde\xi,\tilde\sigma)\in N^\delta_{P(v,\eta)}(\Sigma\times\{0\})$, there exists $\tilde\delta>0$ such that $\|(\xi,\sigma)\|_{L^2\times\mathbb{R}}+\tilde\delta\|(\tilde\xi,\tilde\sigma)\|_{L^2\times\mathbb{R}}<\delta$. Define a curve $\tilde\gamma$

$$
\begin{aligned}
&\tilde\gamma:(-\tilde\delta,\tilde\delta)\to\mathscr{N}^\delta(\Sigma),\\
&\tilde\gamma(s):=\Phi(P(v,\eta),(\xi+s\tilde\xi,\sigma+s\tilde\sigma)).
\end{aligned}
$$

Note that for this curve we have

$$
\begin{aligned}
\tilde\gamma(0)&=\Phi(P(v,\eta),(\xi,\sigma))=(v,\eta),\\
\tilde\gamma'(0)&=Pt^\gamma_{P(v,\eta)}(\tilde\xi,\tilde\sigma).
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
P(\tilde\gamma(s))&=P\circ\Phi(P(v,\eta),(\xi+s\tilde\xi,\sigma+s\tilde\sigma))=P(v,\eta),\qquad\forall\ s\in(-\tilde\delta,\tilde\delta),\\
dP_{(v,\eta)}\circ Pt^\gamma_{P(v,\eta)}(\tilde\xi,\tilde\sigma)&=dP_{(v,\eta)}(\tilde\gamma'(0))=\frac{d}{ds}\Big|_{s=0}P(\tilde\gamma(s))=\frac{d}{ds}\Big|_{s=0}P(v,\eta)=0.
\end{aligned}
$$

□

3.6.2 Taylor expansion of the action functional

In the next subsection we will analyze the behavior of Floer trajectories in the neighborhood $\mathscr{N}(\Sigma \times \{0\})$ of the critical set $\Sigma \times \{0\}$. In particular we will analyze the image of the Floer trajectories under the projection P. In this analysis we will use the Taylor expansion of $\nabla^{J_0}\mathcal{A}^H(v,\eta)$ for $(v,\eta) \in \mathscr{N}^\delta(\Sigma)$ with respect to $P(v,\eta)$.

Following Lang in [25], we define the Taylor expansion of $\nabla^{J_0}\mathcal{A}^H(v,\eta)$ for (v,η) with respect to $P(v,\eta)$ and formulate its convergence as follows

$$\nabla^{J_0}\mathcal{A}^H(v,\eta) = Pt^{\gamma}_{(P(v,\eta))}\big(\sum_{k=0}^{m}\frac{1}{k!}(D_{\gamma'})^k\nabla^{J_0}\mathcal{A}^H(P(v,\eta))\big) + \mathcal{O}(\Phi^{-1}(v,\eta))$$

$$\text{where} \qquad \lim_{\|\Phi^{-1}(\eta,\nabla)\|_{g_{J_0}}\to 0}\frac{\|\mathcal{O}(\Phi^{-1}(v,\eta))\|_{g_{J_0}}}{\|\Phi^{-1}(v,\eta)\|^{m+1}_{g_{J_0}}} < +\infty,$$

where $D_{\gamma'}$ stands for the covariant derivative along γ.

In [25] Serge Lang proves the local convergence of the Taylor series in the general setting of infinite dimensional Riemannian manifolds. However, for further purposes, we are only interested in the Taylor expansion of $\nabla\mathcal{A}^H(v,\eta)$ up to the first order, but we would like it to be uniformly convergent to the parallel transport of its Taylor expansion at $P(v,\eta)$ on the whole $\mathscr{N}^\delta(\Sigma)$. In other words we would like to prove the following lemma:

Lemma 3.13. *Assume* $\sup_{\mathbb{R}^{2n}}\|HessH\| < M$. *Then for all* $(v,\eta) \in \mathscr{N}^\delta(\Sigma)$ *the following holds*

$$\nabla\mathcal{A}^H(v,\eta) = Pt^{\gamma}\big(\nabla\mathcal{A}^H(P(v,\eta)) + \nabla^2_{P(v,\eta)}\mathcal{A}^H(\Phi^{-1}(v,\eta))\big) + \mathcal{O}(\Phi^{-1}(v,\eta)),$$

$$\textit{where} \qquad \|\mathcal{O}(\Phi^{-1}(v,\eta))\|_{L^2\times\mathbb{R}} \le \frac{1}{2}M\|\Phi^{-1}(v,\eta)\|^2_{L^2\times\mathbb{R}}.$$

In particular, this holds for H satisfying condition (H2).

Proof. Let us try to estimate the rest of Taylor expansion near the critical hypersurface $\Sigma \times \{0\}$. Fix $(v,\eta) \in \mathscr{N}^\delta(\Sigma) \times \mathbb{R}$. Then there exist

$$\begin{aligned} \bar v \in \Sigma \quad &\text{such that} \quad P(v,\eta) = (\bar v, 0),\\ \text{and} \quad (\xi,\sigma) \in N^\delta_{P(v,\eta)}(\Sigma\times\{0\}) \quad &\text{such that} \quad \Phi^{-1}(v,\eta) = (\xi,\sigma) = (v-\bar v,\eta). \end{aligned}$$

Then we can estimate the rest of the Taylor expansion by

$$\begin{aligned} \mathcal{O}(\Phi^{-1}(v,\eta)) &= \nabla\mathcal{A}^H_{(v,\eta)} - Pt^{\gamma}(\nabla\mathcal{A}^H_{P(v,\eta)} + \nabla^2_{P(v,\eta)}\mathcal{A}^H(\xi,\eta))\\ &= \begin{pmatrix} -J_0(\partial_t v - \eta X^H(v)) \\ -\int H(v) \end{pmatrix} - \begin{pmatrix} -J_0(\partial_t \xi - \eta X^H(\bar v)) \\ -\int dH_{\bar v}(\xi) \end{pmatrix}\\ &= \begin{pmatrix} -J_0(\partial_t v - \eta X^H(v) - \partial_t(v-\bar v) + \eta X^H(\bar v)) \\ -\int(H(v) - dH_{\bar v}(\xi)) \end{pmatrix}\\ &= \begin{pmatrix} \eta J_0(X^H(v) - X^H(\bar v)) \\ -\int(H(v) - dH_{\bar v}(\xi)) \end{pmatrix}. \end{aligned}$$

Since the Hessian of H is uniformly bounded, we obtain

$$\begin{aligned}\|\eta J_0(X^H(v) - X^H(\bar{v}))\|_{L^2} &\leq |\eta|\|\nabla H(v) - \nabla H(\bar{v})\|_{L^2} \\ &\leq |\eta|\|v - \bar{v}\|_{L^2}\Big(\int \sup_{s\in[0,1]} \|Hess_{\bar{v}+s(v(t)-\bar{v})}H\|^2 dt\Big)^{\frac{1}{2}} \\ &\leq M|\eta|\|\xi\|_{L^2}.\end{aligned}$$

Analogically using once more the Taylor expansion for H at $\bar{v}$ we obtain

$$\begin{aligned}\Big|\int (H(v(t)) - dH_{\bar{v}}(v(t) - \bar{v}))dt\Big| &\leq \int \frac{1}{2} \sup_{s\in[0,1]} \|Hess_{\bar{v}+s(v(t)-\bar{v})}H\|\|v(t) - \bar{v}\|^2 dt \\ &\leq \frac{1}{2}M\|\xi\|^2_{L^2}.\end{aligned}$$

The two results combined give

$$\begin{aligned}\|\mathcal{O}(\Phi^{-1}(v,\eta))\|_{L^2\times\mathbb{R}} &= \|\nabla\mathcal{A}^H_{(v,\eta)} - Pt^\gamma(\nabla\mathcal{A}^H_{P(v,\eta)} + \nabla^2_{P(v,\eta)}\mathcal{A}^H(\xi,\eta))\|_{L^2\times\mathbb{R}} \\ &\leq \frac{1}{2}M(\|\xi\|_{L^2} + |\eta|)^2.\end{aligned}$$

□

3.6.3 Properties of the Hessian of the action functional

In this subsection we will prove some properties of the Hessian of the action functional both on $\Sigma\times\{0\}$ and in its tubular neighborhood. We will use these properties later in the analysis of the projection of the Floer trajectory and in the proof of Proposition 3.16.

Lemma 3.14. *For all* $(v,\eta) \in \mathscr{N}^\delta(\Sigma)$

$$\nabla^2_{P(v,\eta)}\mathcal{A}^H(\Phi^{-1}(v,\eta)) \in N_{P(v,\eta)}(\Sigma\times\{0\}).$$

Proof. Recall that we identify the normal bundle of $\Sigma\times\{0\}$, using the metric g_{J_0}, with

$$N_{(v,0)}(\Sigma\times\{0\}) = \{\xi \in C^\infty(S^1, v^*T\mathbb{R}^{2n}) \mid \langle \int_0^1 \xi(t)dt, \hat{\xi}\ \rangle = 0,\ \forall\ \hat{\xi} \in T_vH^{-1}(0)\} \times \mathbb{R}.$$

On the other hand, the Hessian of $\mathcal{A}^H$ viewed as a linear map

$$\nabla^2_{(v,\eta)}\mathcal{A}^H : T_vC^\infty(S^1,\mathbb{R}^{2n})\times\mathbb{R} \to T_vC^\infty(S^1,\mathbb{R}^{2n})\times\mathbb{R},$$

of the form

$$\nabla^2_{(v,\eta)}\mathcal{A}^H(\xi,\sigma) = \begin{pmatrix} -J_0(\partial_t\xi - \sigma X^H(v)) - \eta\ Hess_vH(\xi) \\ -\int dH(\xi) \end{pmatrix}.$$

Take $(v, \eta) \in \mathscr{N}^\delta(\Sigma)$. Then by definition there exists $\bar{v} \in \Sigma$ and $(\xi, \sigma) \in N^\delta_{(\bar{v},0)}(\Sigma \times \{0\})$ such that

$$P(v, \eta) = (\bar{v}, 0), \quad \text{and} \quad \Phi((\bar{v}, 0), (\xi, \sigma)) = (v, \eta).$$

In particular $\sigma = \eta$.

In view of $N(\Sigma \times \{0\})$ and $\nabla^2 \mathcal{A}^H$ stated above, to prove the lemma it is enough to show that for all $\hat{\xi} \in T_{\bar{v}} H^{-1}(0)$

$$\langle \int J_0(\partial_t \xi - \eta X^H(\bar{v}))dt, \hat{\xi} \rangle = 0.$$

But this is an easy consequence of the fact that

$$\int \partial_t \xi dt = 0 \qquad \text{and} \qquad - J_0 X^H(\bar{v}) = \nabla H(\bar{v}).$$

□

Recall that by Lemma 2.23, we have that for $v \in \Sigma$, $Ker(\nabla^2_{(v,0)} \mathcal{A}^H) = T_v \Sigma \times \{0\}$. As a consequence the restriction of $\nabla^2_{(v,0)} \mathcal{A}^H$ to the complement of the kernel is injective. In the following Lemma we will prove even stronger property estimating the norm of the inverse.

Lemma 3.15. *If $v \in \Sigma$ and $\|\nabla H(v)\| \geq \frac{1}{2}$, then for $(\xi, \eta) \in N_{(v,0)}(\Sigma \times \{0\})$*

$$\|\nabla^2_{(v,0)} \mathcal{A}^H(\xi, \eta)\|_{L^2 \times \mathbb{R}} \geq \frac{1}{6}(\|\xi\|_{W^{1,2}} + |\eta|).$$

Moreover, whenever $\|\nabla H(v)\| \geq 1$ then in fact for all $(\xi, \eta) \in N_{(v,0)}(\Sigma \times \{0\})$

$$\|\nabla^2_{(v,0)} \mathcal{A}^H(\xi, \eta)\|_{L^2 \times \mathbb{R}} \geq \frac{1}{3}(\|\xi\|_{L^2} + |\eta|).$$

Proof. By definition

$$N_{(v,0)}(\Sigma \times \{0\}) = \{\xi \in C^\infty(S^1, v^*T\mathbb{R}^{2n}) \mid \langle \int_0^1 \xi(t)dt, \hat{\xi} \rangle = 0, \ \forall\, \hat{\xi} \in T_v H^{-1}(0)\} \times \mathbb{R},$$

hence for $(\xi, \eta) \in N_{(v,0)}(\Sigma \times \{0\})$ one has

$$\begin{aligned} \int_0^1 \xi(t)dt &= \frac{\langle \int \xi(t)dt, \nabla H(v) \rangle}{\|\nabla H(v)\|^2} \nabla H(v) \\ &= \frac{\int dH_v(\xi)dt}{\|\nabla H(v)\|^2} \nabla H(v), \\ \Big| \int dH_v(\xi)dt \Big| &= \Big\| \int_0^1 \xi(t)dt \Big\| \, \|\nabla H(v)\|. \end{aligned}$$

Taking the above equality into account and recalling that for $\xi \in W^{1,2}(S^1, \mathbb{R}^{2n})$

$$\|\partial_t \xi\|_{L^2} + \left\| \int \xi(t)dt \right\| \geq \|\xi\|_{L^2},$$

we can bound the Hessian of $\mathcal{A}^H$ form below in the following way

$$\begin{aligned}
\|\nabla^2_{(v,0)}\mathcal{A}^H(\xi,\eta)\|_{L^2\times\mathbb{R}} &= \|\partial_t\xi - \eta X^H(v)\|_{L^2} + \left| \int dH_v(\xi) \right| \\
&= \|\partial_t\xi - \eta X^H(v)\|_{L^2} + \left\| \int \xi(t)dt \right\| \, \|\nabla H(v)\| \\
&\geq \frac{1}{3}(\|\partial_t\xi\|_{L^2} - |\eta|\|X^H(v)\|_{L^2}) + \frac{2}{3}\left\| \int (\partial_t\xi - \eta X^H(v))dt \right\| + \left\| \int \xi(t)dt \right\| \|\nabla H(v)\| \\
&= \frac{1}{3}\|\partial_t\xi\|_{L^2} + \frac{1}{3}|\eta|\|X_H(v)\|_{L^2} + \left\| \int \xi(t)dt \right\| \, \|\nabla_v H\| \\
&\geq \frac{1}{6}\|\partial_t\xi\|_{L^2} + \frac{1}{6}\|\xi\|_{L^2} + \frac{1}{3}|\eta|\|\nabla H(v)\|_{L^2} + \left\| \int \xi(t)dt \right\| \left(\|\nabla H(v)\| - \frac{1}{6} \right) \\
&\geq \frac{1}{6}(\|\xi\|_{W^{1,2}} + |\eta|)
\end{aligned}$$

The last inequality comes from the assumption that $\|\nabla H(v)\| \geq \frac{1}{2}$. This proves the injectivity and the first inequality.

For the second one we calculate similarly

$$\begin{aligned}
\|\nabla^2_{(v,0)}\mathcal{A}^H(\xi,\eta)\|_{L^2\times\mathbb{R}} &= \|\partial_t\xi - \eta X^H(v)\|_{L^2} + \left| \int dH_v(\xi) \right| \\
&= \|\partial_t\xi - \eta X^H(v)\|_{L^2} + \left\| \int \xi(t)dt \right\| \, \|\nabla H(v)\| \\
&\geq \frac{1}{3}(\|\partial_t\xi\|_{L^2} - |\eta|\|X^H(v)\|_{L^2}) + \frac{2}{3}\left| \int (\partial_t\xi - \eta X^H(v))dt \right| + \left\| \int \xi(t)dt \right\| \, \|\nabla_v H\| \\
&= \frac{1}{3}\|\partial_t\xi\|_{L^2} + \frac{1}{3}|\eta|\|X^H(v)\|_{L^2} + \left\| \int \xi(t)dt \right\| \, \|\nabla H(v)\| \\
&\geq \frac{1}{3}\|\xi\|_{L^2} + \frac{1}{3}|\eta|\|\nabla H(v)\|_{L^2} + \left\| \int \xi(t)dt \right\| \left(\|\nabla H(v)\| - \frac{1}{3} \right) \\
&\geq \frac{1}{3}(\|\xi\|_{L^2} + |\eta|),
\end{aligned}$$

where the last inequality holds provided $\|\nabla H(v)\| \geq 1$. □

3.6.4 Projecting a Floer trajectory

Let $u : \mathbb{R} \to C^\infty(S^1, \mathbb{R}^{2n}) \times \mathbb{R}$ be a Floer trajectory, i.e. $u(s) = (v(s,t), \eta(s))$ and

$$\partial_s u = \begin{pmatrix} \partial_s v \\ \partial_s \eta \end{pmatrix} = \begin{pmatrix} -J(\partial_t v - \eta X^H(v)) \\ -\int H(v) \end{pmatrix}.$$

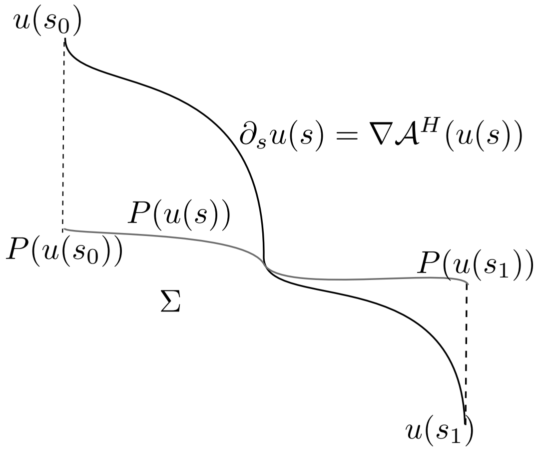

Figure 3.1: Schematic depiction of the Floer trajectory near the critical set $\Sigma \times \{0\}$.

Let us analyze $P(u(s))$. Can we bound the distance $P(u(s))$ travels along the hypersurface Σ for $u(s) \in \mathscr{N}(\Sigma)$?

To simplify the notation we will introduce the set $\mathscr{N}_r^\delta(\Sigma)$. Recall the tubular neighborhood $\mathscr{N}^\delta(\Sigma)$ defined in (3.26). Then for small enough $\delta > 0$ and any $r > 0$ we will denote

$$\mathscr{N}_r^\delta(\Sigma) := \{x \in \mathscr{N}^\delta(\Sigma) \mid \|P(x)\| \geq r\} \tag{3.27}$$

Observe that even though the definition of $\mathcal{U}_\delta^0$ depends on whether we take Σ to be $H_0^{-1}(0)$ or $H_1^{-1}(0)$, however for $r > \sup_{x\in K}\|x\|$ the definition of $\mathscr{N}_r^\delta(\Sigma)$ does not depend on the choice of Σ, i.e. :

$$\mathscr{N}_r^\delta(H_0^{-1}(0)) = \mathscr{N}_r^\delta(H_1^{-1}(0)),$$

since H_0 and H_1 differ only on the compact set K. Moreover, we have

$$\{(v,\eta) \in \mathcal{U}_\delta^0 \mid \|v\|_{L^2} \geq r + \delta\} \subseteq \mathscr{N}_r^\delta(\Sigma).$$

Proposition 3.16. *Assume H satisfies conditions (H1), (H2) and (H3). Then there exists $\delta > 0$ and $r > 0$, such that if u is a Floer trajectory*

$$u : \mathbb{R} \to C^\infty(S^1, \mathbb{R}^{2n}) \times \mathbb{R}$$
$$u(s) = (v(s), \eta(s)),$$

and $s_0, s_1 \in \mathbb{R}$ are such that

$$\forall\ s \in [s_0, s_1] \qquad u(s) \in \mathscr{N}_r^\delta(\Sigma),$$

the following holds true

$$\|P(u(s_1)) - P(u(s_0))\| \leq \hat{M}|\mathcal{A}^H(u(s_1)) - \mathcal{A}^H(u(s_0))|,$$

where $\hat{M} = 18M$ and $M = \sup_{x\in\mathbb{R}^{2n}}\|Hess_x H\|$ is finite by (H2).

Proof. Define

$$r := \frac{c_1 + \sqrt{c_1^2 + 4c_2(c_3 + c_1)}}{2c_2}. \tag{3.28}$$

Then for any $(v, \eta) \in \mathscr{N}_r^{\delta}(\Sigma)$ we have

$$\begin{aligned} \frac{c_1 + \sqrt{c_1^2 + 4c_2(c_3 + c_1)}}{2c_2} &\leq \|P(v, \eta)\|, \\ c_2\|P(v, \eta)\|^2 - c_1\|P(v, \eta)\| - (c_3 + c_1) &\geq 0, \\ \frac{c_2\|P(v, \eta)\|^2 - c_3}{c_1(\|P(v, \eta)\| + 1)} &\geq 1. \end{aligned}$$

Recall that from the assumptions on the Hamiltonian if follows

$$c_2\|P(v, \eta)\|^2 - c_3 \leq dH_{P(v,\eta)}(X^{\dagger}) \leq \|\nabla H(P(v, \eta)\|c_1(\|P(v, \eta)\| + 1).$$

All combined it guarantees that whenever

$$(v, \eta) \in \mathscr{N}_r^{\delta}(\Sigma),$$

then in fact

$$\|\nabla H(P(v, \eta))\| \geq 1$$

Moreover, recall from Lemma 3.11 that $\mathscr{U}_{\delta}^0 = \mathscr{N}^{\delta}(\Sigma)$.

That means that whenever $u(s) \in \mathscr{N}_r^{\delta}(\Sigma)$ for all $s \in [s_0, s_1]$ then the assumptions of Lemma 3.15 are satisfied.

Now, knowing that $\partial_s u = \nabla \mathcal{A}^H(u)$ and using the Taylor expansion for $\nabla \mathcal{A}^H$ from Lemma 3.13 we obtain

$$\begin{aligned} P(u(s_1)) - P(u(s_0)) &= \int_{s_0}^{s_1} \frac{d}{ds} P(u(s)) ds \\ &= \int_{s_0}^{s_1} dP_{u(s)} \partial_s u(s) ds \\ &= \int_{s_0}^{s_1} dP_{u(s)} \nabla \mathcal{A}^H(u(s)) ds \\ &= \int_{s_0}^{s_1} dP_{u(s)} \Big(Pt^{\gamma} \big(\nabla \mathcal{A}^H(P(u(s))) + \nabla^2_{P(u(s))} \mathcal{A}^H(\Phi^{-1}(u(s))) \big) + \mathcal{O}(\Phi^{-1}(u(s)) \Big) ds \\ &= \int_{s_0}^{s_1} dP_{u(s)} \Big(Pt^{\gamma} \big(\nabla^2_{P(u(s))} \mathcal{A}^H(\Phi^{-1}(u(s))) \big) + \mathcal{O}(\Phi^{-1}(u(s)) \Big) ds. \end{aligned}$$

Combining results from Lemma 3.12 and Lemma 3.14 we obtain that for all $(v, \eta) \in \mathscr{N}^{\delta}(\Sigma)$

$$\nabla^2_{P(v,\eta)} \mathcal{A}^H(\Phi^{-1}(v, \eta)) \in N_{P(v,\eta)}(\Sigma \times \{0\}) \subseteq Ker(dP_{(v,\eta)} \circ Pt^{\gamma}_{P(v,\eta)}),$$

hence the first expression under the integral in fact vanishes.

Now we will use the estimate on $\mathcal{O}$ from Lemma 3.13 to get

$$\begin{aligned}\|P(u(s_1)) - P(u(s_0))\| &\leq \int_{s_0}^{s_1} \|dP_{u(s)}\mathcal{O}(\Phi^{-1}(u(s)))\|^2_{L^2\times\mathbb{R}} ds \\ &\leq \int_{s_0}^{s_1} \|\mathcal{O}(\Phi^{-1}(u(s)))\|^2_{L^2\times\mathbb{R}} ds \\ &\leq \frac{1}{2}M \int_{s_0}^{s_1} \|\Phi^{-1}(u(s))\|^2_{L^2\times\mathbb{R}} ds.\end{aligned}$$

Let us now make a change of variables. Note that

$$\begin{aligned}\Psi : [s_0, s_1] &\to [\mathcal{A}^H(u(s_0)), \mathcal{A}^H(u(s_1))], \\ \Psi(s) &:= \mathcal{A}^H(u(s)), \\ \Psi'(s) &= \|\nabla\mathcal{A}^H u(s)\|^2_{L^2\times\mathbb{R}},\end{aligned}$$

is in fact a diffeomorphism. Substituting into the equation we get

$$\|P(u(s_1)) - P(u(s_0))\| \leq \frac{1}{2}M \int_{\mathcal{A}^H(u(s_0))}^{\mathcal{A}^H(u(s_1))} \frac{\|\Phi^{-1} \circ u \circ \Psi^{-1}(\tau)\|^2_{L^2\times\mathbb{R}}}{\|\nabla\mathcal{A}^H(u \circ \Psi^{-1}(\tau))\|^2_{L^2\times\mathbb{R}}} d\tau.$$

Let us now estimate the denominator under the integral. For $u \in \mathscr{N}(\Sigma)$ such that $\|\nabla H(P(u))\| \geq 1$, we can use the result from Lemma 3.15

$$\|\nabla^2_{P(u)}\mathcal{A}^H(\Phi^{-1}(u))\|_{L^2\times\mathbb{R}} \geq \frac{1}{3}\|\Phi^{-1}(u)\|_{L^2\times\mathbb{R}},$$

and the estimate from Lemma 3.13 to calculate

$$\begin{aligned}\|\nabla\mathcal{A}^H(u)\|_{L^2\times\mathbb{R}} &\geq \|\nabla^2_{P(u)}\mathcal{A}^H(\Phi^{-1}(u))\|_{L^2\times\mathbb{R}} - \frac{1}{2}M\|\Phi^{-1}(u)\|^2_{L^2\times\mathbb{R}} \\ &\geq \frac{1}{3}\|\Phi^{-1}(u)\|_{L^2\times\mathbb{R}} - \frac{1}{2}M\|\Phi^{-1}(u)\|^2_{L^2\times\mathbb{R}} \\ &= \|\Phi^{-1}(u)\|_{L^2\times\mathbb{R}}(\frac{1}{3} - \frac{1}{2}M\|\Phi^{-1}(u)\|_{L^2\times\mathbb{R}}).\end{aligned}$$

Therefore, for

$$\|\Phi^{-1}(u)\|_{L^2\times\mathbb{R}} \leq \frac{1}{3M},$$

one has

$$\|\nabla\mathcal{A}^H(u)\|_{L^2\times\mathbb{R}} \geq \frac{1}{6}\|\Phi^{-1}(u)\|_{L^2\times\mathbb{R}}.$$

Let $\tilde{\delta}(H) > 0$ be a constant as in Lemma 4.14. If we now take $\delta > 0$, such that

$$\delta < \min\{\frac{1}{3M}, \tilde{\delta}(H)\}$$

then by (3.26) and Lemma 4.14 the projection $\mathscr{N}(\Sigma) \to \Sigma$ is well defined and we can combine the two results above to analyze the expression under the integral

$$\frac{\|\Phi^{-1}(u)\|^2_{L^2\times\mathbb{R}}}{\|\nabla\mathcal{A}^H(u)\|^2_{L^2\times\mathbb{R}}} \leq \frac{6^2\|\Phi^{-1}(u)\|^2_{L^2\times\mathbb{R}}}{\|\Phi^{-1}(u)\|^2_{L^2\times\mathbb{R}}} = 6^2.$$

Therefore, taking $\hat{M} = 18M$, we obtain

$$\begin{aligned}\|P(u(s_1)) - P(u(s_0))\| &\leq \frac{1}{2}M\int_{\mathcal{A}^H(u(s_0))}^{\mathcal{A}^H(u(s_1))} \frac{\|\Phi^{-1}\circ u\circ\Psi^{-1}(\tau)\|^2_{L^2\times\mathbb{R}}}{\|\nabla\mathcal{A}^H(u\circ\Psi^{-1}(\tau))\|^2_{L^2\times\mathbb{R}}}d\tau\\ &\leq \hat{M}|\mathcal{A}^H(u(s_1)) - \mathcal{A}^H(u(s_0))|,\end{aligned}$$

as claimed. □

3.7 Oscillations and L^2 bounds

The goal of this section to show that for a homotopy Γ satisfying assumptions of Theorem 7, if we fix $a, b \in \mathbb{R}$ and a compact subset N, such that $N_0 \subseteq N \subseteq H_0^{-1}(0)$, then for each pair (Λ_0, Λ_1) of connected components

$$\begin{aligned}\Lambda_0 &\subseteq \mathscr{C}(\mathcal{A}^{H_0}, N) \cap (\mathcal{A}^{H_0})^{-1}([a,\infty))\\ \Lambda_1 &\subseteq Crit(\mathcal{A}^{H_1}) \cap (\mathcal{A}^{H_1})^{-1}((-\infty, b])\end{aligned}$$

all the Floer trajectories in the associated moduli space $\mathscr{M}^\Gamma(\Lambda_0,\Lambda_1) \subseteq C^\infty(\mathbb{R}\times S^1, \mathbb{R}^{2n}\times\mathbb{R})$ are uniformly bounded in the $L^2\times\mathbb{R}$ norm.

Recall that in Proposition 3.4 we have already established bounds on the η component. To establish the L^2 bounds on the v component of a Floer trajectory one has to analyze the Floer trajectory not only outside of the set of infinitesimal action derivation, but also estimate how far it travels within $\mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon)$, along the hypersurface $\Sigma\times\{0\}$. To obtain the uniform bounds, we cover the space $C^\infty(S^1,\mathbb{R}^{2n})\times\mathbb{R}$ by three sets on which the Floer trajectory can be bounded differently. The precise construction of these sets is carried out in the proof of Theorem 7. In particular, if we choose $\delta \in (0,\mathfrak{y})$ small enough to satisfy the assumptions of Proposition 3.16, and $\varepsilon \in (0, \varepsilon_2(\delta/2, \|J\|_\infty))$ as in Proposition 3.8, then we have the following inclusions:

$$\mathcal{B}^J(\mathfrak{a},\mathfrak{y},\varepsilon) \subseteq \mathcal{K}^1_{\delta/2}\cup\mathcal{U}^1_{\delta/2} \subseteq \mathcal{K}^0_\delta\cup\mathcal{U}^0_{\delta/2} \subseteq \mathcal{K}^0_\delta\cup\mathcal{U}^0_\delta,$$

where the sets have been defined in Definition 3.7, (3.24), (3.22), (3.11) and (3.25). The reason why we choose here $\frac{1}{2}\delta$ instead of δ is to estimate the number of oscillations of the Floer trajectory and will became apparent later in the proof of Lemma 3.17.

Consider the set $\mathscr{N}^\delta_{\mathrm{v}_4}$ defined as in (3.27) for v_4 from (3.10). Since $\mathrm{v}_4 \geq \sup_{x\in K}\|x\|$ therefore $\mathscr{N}^\delta_{\mathrm{v}_4}$ does not depend on which H_s we choose to define it with. Moreover, we obtain the following inclusions:

$$\begin{aligned}\mathcal{U}^0_\delta \setminus \mathcal{K}^0_\delta &\subseteq \mathscr{N}^\delta_{\mathrm{v}_4},\\ \mathcal{B}^J(\mathfrak{a},\mathfrak{y},\varepsilon) \subseteq \mathcal{K}^0_\delta\cup\mathcal{U}^0_\delta &= \mathcal{K}^0_\delta\cup\mathscr{N}^\delta_{\mathrm{v}_4},\end{aligned}$$

and thus we can cover the space $C^\infty(S^1,\mathbb{R}^{2n})\times\mathbb{R}$ with the following sets

$$\mathcal{K}^0_\delta, \qquad \mathscr{N}^\delta_{\mathrm{v}_4}, \qquad (C^\infty(S^1,\mathbb{R}^{2n})\times\mathbb{R})\setminus\mathcal{B}^J(\mathfrak{a},\mathfrak{y},\varepsilon),$$

and, as mentioned before, on each of them the Floer trajectory can be bounded differently.

The time a Floer trajectory spends outside $\mathcal{B}^J(\mathfrak{a},\mathfrak{y},\varepsilon)$ is limited therefore its growth in the $L^2\times\mathbb{R}$ norm can be estimated using results from Lemma 3.17. In a similar way one can bound the growth of the $L^2\times\mathbb{R}$ norm on the interval $[0,1]$ where the Hamiltonian might change, but the time is limited. However, the time a Floer trajectory spends in $\mathcal{B}^J(\mathfrak{a},\mathfrak{y},\varepsilon)$ is not limited. If $s\notin(0,1)$ then (H_s,J_s) is either then (H_s,J_s) is either $(H_0,\{J_{0,t}(\cdot,\eta)\}_{(t,\eta)\in S^1\times\mathbb{R}})$ or $(H_1,\{J_{1,t}(\cdot,\eta)\}_{(t,\eta)\in S^1\times\mathbb{R}})$. Moreover, since we have chosen $\delta\in(0,\mathfrak{y})$ this guarantees that all

$$J_s\in\mathscr{J}\left(\mathbb{R}^{2n},\omega_0,\mathcal{V}\times\left((-\infty,\mathfrak{y})\cup(\mathfrak{y},\infty)\right)\right)$$

will be constant and equal J_0 on $\mathscr{N}^\delta_{\mathrm{v}_4}$. Finally, $\mathrm{v}_4\geq r$, where r is as in (3.28) and thus on $\mathscr{N}^\delta_{\mathrm{v}_4}$ we can apply Proposition 3.16 directly, which gives the bounds on the growth of the Floer trajectory in the $L^2\times\mathbb{R}$ norm outside of the interval $(0,1)$ and in $\mathscr{N}^\delta_{\mathrm{v}_4}$.

To obtain the uniform bounds for the whole Floer trajectory, we will have to determine how it oscillates between the two non-compact sets, i.e. between $\mathcal{B}^J(\mathfrak{a},\mathfrak{y},\varepsilon)$ and $(C^\infty(S^1,\mathbb{R}^{2n})\times\mathbb{R})\setminus\mathscr{N}^\delta_{\mathrm{v}_4}$.

Let us consider setting as in Theorem 7 and a pair of connected components (Λ_0,Λ_1) satisfying

$$\begin{aligned}\Lambda_0&\subseteq\mathscr{C}(\mathcal{A}^{H_0},N)\cap(\mathcal{A}^{H_0})^{-1}([a,\infty))\\ \Lambda_1&\subseteq Crit(\mathcal{A}^{H_1})\cap(\mathcal{A}^{H_1})^{-1}((-\infty,b])\\ &\mathscr{M}^\Gamma(\Lambda_0,\Lambda_1)\neq\emptyset\end{aligned}$$

for a fixed pair of $a,b\in\mathbb{R}$. By Lemma 3.9 one can assume

$$\Lambda_0\subseteq\mathcal{K}^1_{\delta/2}\subseteq\mathcal{K}^0_\delta,$$

with the sets $\mathcal{K}^1_{\delta/2},\mathcal{K}^0_\delta$ defined as in (3.24) and (3.11), respectively.

Take a Floer trajectory $u\in\mathscr{M}^\Gamma(\Lambda_0,\Lambda_1)$, and fix $s\in\mathbb{R}$. For such s let us define a sequence in $\mathbb{R}$

$$\tau_1(s)\leq\tau_2^+(s)\leq\tau_2^-(s)\leq\tau_3^+(s)\cdots\leq s$$

in the following way

$$\begin{aligned}\tau_1(s)&:=\sup\{\tau\leq s\mid u(\tau)\in\mathcal{K}^0_\delta\},\\ \tau_2^+(s)&:=\inf\{\tau_1(s)\leq\tau\leq s\mid u(\tau)\notin\mathcal{K}^0_\delta\cup\mathscr{N}^\delta_r\},\\ \tau_k^-(s)&:=\inf\{\tau_k^+(s)\leq\tau\leq s\mid u(\tau)\in\mathcal{B}^J(\mathfrak{a},\mathfrak{y},\varepsilon)\},\\ \tau_{k+1}^+(s)&:=\inf\{\tau_k^-(s)\leq\tau\leq s\mid u(\tau)\notin\mathcal{K}^0_\delta\cup\mathscr{N}^\delta_r\}.\end{aligned}$$

We stop the sequence if the set on the right becomes empty. Note that by assumption

$$\lim_{s\to-\infty} u(s) \in \Lambda_0 \subseteq \mathcal{K}^1_{\delta/2}$$

and therefore $\tau_1(s)$ is well defined and finite. This means that the sequence has always at least one element.

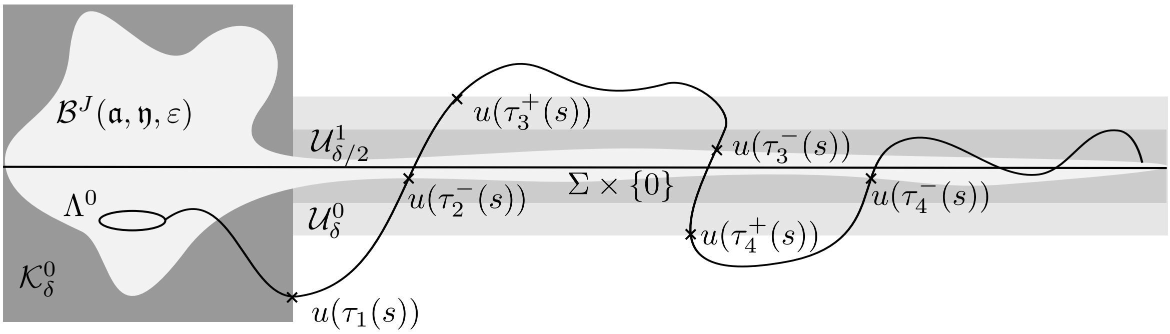

Figure 3.2: Construction of the sequence $\tau_1(s) \leq \tau_2^+(s) \leq \tau_2^-(s) \leq \tau_3^+(s) \ldots$. Here Λ^0 is a connected component of $\mathscr{C}(\mathcal{A}^{H_0}, N_0) \cap (\mathcal{A}^{H_0})^{-1}([a,\infty))$.

By definition the sequence satisfies the following properties

$$\forall\, \tau \in (\tau_1(s), \tau_2^+(s)), \qquad u(\tau) \in \mathscr{N}_r^\delta, \tag{3.29}$$

$$\forall\, \tau \in (\tau_k^+(s), \tau_k^-(s)), \qquad u(\tau) \notin \mathcal{B}^J(\mathfrak{a},\mathfrak{y},\varepsilon), \tag{3.30}$$

$$\forall\, \tau \in (\tau_k^-(s), \tau_{k+1}^+(s)), \qquad u(\tau) \in \mathscr{N}_r^\delta. \tag{3.31}$$

This way the sequence carries some information on where the Floer trajectory is: in $\mathscr{N}_r^\delta$ or in $(C^\infty(S^1,\mathbb{R}^{2n})\times\mathbb{R}) \setminus \mathcal{B}^J(\mathfrak{a},\mathfrak{y},\varepsilon)$ (it can be in both, thought). The length of the sequence indicates how many times the Floer trajectory oscillates between the two sets.

Having the sequence defined as above we will call an oscillation a connected component of the Floer trajectory of the form

$$u^{-1}([\tau_k^+(s), \tau_{k+1}^+(s)]),$$

for some $1 \leq k \in \mathbb{N}$. In the following lemma we will prove that the sequence is in fact finite, thus the number of oscillations is finite, too.

Lemma 3.17. *Let $\Gamma = \{H_s, J_s\}_{s\in\mathbb{R}}$ be a smooth homotopy of Hamiltonians and almost complex structures defined as in Theorem 7. Fix a pair $a, b \in \mathbb{R}$ and two connected components*

$$\Lambda_0 \subseteq \mathscr{C}(\mathcal{A}^{H_0}, N) \cap (\mathcal{A}^{H_0})^{-1}([a,\infty)),$$
$$\Lambda_1 \subseteq Crit(\mathcal{A}^{H_1}) \cap (\mathcal{A}^{H_1})^{-1}((-\infty,b]).$$

Let $\mathfrak{e}$ be the bound on the energy of the Floer trajectories of $\mathscr{M}^\Gamma(\Lambda^0,\Lambda^1)$ as provided by Proposition 3.4. Fix $u \in \mathscr{M}^\Gamma(\Lambda^0,\Lambda^1)$ and $s \in \mathbb{R}$. Let

$$\tau_1(s) \leq \tau_2^+(s) \leq \tau_2^-(s) \leq \tau_3^+(s) \cdots \leq s$$

be a sequence associated to s and defined as above. Let $K \in \mathbb{N}$. Then

$$\sum_{k=2}^{K} \|u(\tau_k^-(s)) - u(\tau_k^+(s))\|_{L^2(S^1)\times\mathbb{R}} \leq \frac{\mathfrak{e}}{\varepsilon}. \tag{3.32}$$

Moreover, this implies that the number of oscillations is in fact finite namely

$$K \leq \frac{2\mathfrak{e}}{\delta\varepsilon} + 1. \tag{3.33}$$

Proof. By Proposition 3.4 we have

$$\|\nabla^{J_s}\mathcal{A}^{H_s}(u(s))\|^2_{L^2(\mathbb{R}\times S^1)} \leq \mathfrak{e}$$

In view of that we can estimate the time a Floer trajectory $u \in \mathscr{M}^{\Gamma}(\Lambda^0, \Lambda^1)$ spends outside $\mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon)$. Denote by $\mathbb{1}_{\mathcal{B}^J(\mathfrak{a},\mathfrak{y},\varepsilon)}$ the characteristic function of $\mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon)$, then

$$\begin{aligned} \mathfrak{e} \geq \|\nabla^{J_s}\mathcal{A}^{H_s}(u(s))\|^2_{L^2(\mathbb{R}\times S^1)} &\geq \varepsilon^2 \int_{\mathbb{R}} \big(1 - \mathbb{1}_{\mathcal{B}^J(\mathfrak{a},\mathfrak{y},\varepsilon)}(u(\tau))\big)d\tau \\ \frac{\mathfrak{e}}{\varepsilon^2} \geq \int_{\mathbb{R}} \big(1 - \mathbb{1}_{\mathcal{B}^J(\mathfrak{a},\mathfrak{y},\varepsilon)}(u(\tau))\big)d\tau. \end{aligned} \tag{3.34}$$

By definition

$$\forall\, \tau \in (\tau_k^+(s), \tau_k^-(s)), \qquad u(\tau) \notin \mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon),$$

therefore we can calculate

$$\begin{aligned} &\sum_{k=2}^{K} \|u(\tau_k^+(s)) - u(\tau_k^-(s))\|_{L^2(S^1)\times\mathbb{R}} \leq \\ &\leq \sum_{k=2}^{K} \sqrt{\int_0^1 \Big(\int_{\tau_k^-(s)}^{\tau_k^+(s)} \|\partial_\tau v(\tau,t)\| d\tau\Big)^2 dt + \Big(\int_{\tau_k^-(s)}^{\tau_k^+(s)} |\partial_\tau \eta(\tau)| d\tau\Big)^2} \\ &\leq \sum_{k=2}^{K} \sqrt{\int_0^1 \int_{\tau_k^+(s)}^{\tau_k^-(s)} 1 d\tau \int_{\tau_k^-(s)}^{\tau_k^+(s)} (\|\partial_\tau v(\tau,t)\|^2 + |\partial_\tau \eta(\tau)|^2) d\tau dt} \\ &= \sum_{k=2}^{K} \sqrt{\int_{\tau_k^+(s)}^{\tau_k^-(s)} 1 d\tau \int_{\tau_k^-(s)}^{\tau_k^+(s)} \|\nabla^{J_\tau}\mathcal{A}^{H_\tau}(u(\tau))\|^2_{L^2(S^1)} d\tau} \\ &\leq \sqrt{\sum_{k=2}^{K} \int_{\tau_k^+(s)}^{\tau_k^-(s)} 1 d\tau} \sqrt{\sum_{k=2}^{K} \int_{\tau_k^-(s)}^{\tau_k^+(s)} \|\nabla^{J_\tau}\mathcal{A}^{H_\tau}(u(\tau))\|^2_{L^2(S^1)} d\tau} \\ &\leq \sqrt{\int_{\mathbb{R}} \big(1 - \mathbb{1}_{\mathcal{B}(a,b,\varepsilon)}(u(\tau))\big) d\tau}\ \|\nabla^{J_\tau}\mathcal{A}^{H_\tau}(u(\tau))\|_{L^2(\mathbb{R}\times S^1)} \\ &\leq \frac{\mathfrak{e}}{\varepsilon} \end{aligned}$$

This proves the first claim.

Due to the fact that $u(s)$ is continuous,

$$u(\tau_k^-(s)) \in cl(\mathcal{B}(a,b,\varepsilon) \setminus \mathcal{K}_\delta^0) \subseteq \mathcal{U}_{\delta/2}^0,$$
$$u(\tau_k^+(s)) \in cl((C^\infty(S^1, \mathbb{R}^{2n}) \times \mathbb{R}) \setminus \mathcal{U}_\delta^0).$$

Following the definition of $\mathcal{U}_\delta^0$, we get that

$$dist_{L^2\times\mathbb{R}}(\mathcal{U}_{\delta/2}^0, cl(C^\infty(S^1,\mathbb{R}^{2n}) \times \mathbb{R} \setminus \mathcal{U}_\delta^0)) = \frac{\delta}{2},$$
$$dist_{L^2\times\mathbb{R}}(u(\tau_k^+(s)), u(\tau_k^-(s))) \geq \frac{\delta}{2}.$$

Combining this with the previous result we obtain the claimed bound on the number of oscillations

$$(K-1)\frac{\delta}{2} \leq \sum_{k=2}^{K} \|u(\tau_k^+(s)) - u(\tau_k^-(s))\|_{L^2\times\mathbb{R}} \leq \frac{\mathfrak{e}}{\varepsilon}$$
$$K \leq \frac{2\mathfrak{e}}{\delta\varepsilon} + 1.$$

□

This proves that the number of oscillations is finite, which will allow us to show the boundedness of v component of $\mathscr{M}^\Gamma(\Lambda^0, \Lambda^1)$ in the L^2 norm. In the following proposition we first prove the $L^2 \times \mathbb{R}$ bounds, which immediately implies the $L^\infty \times \mathbb{R}$ bounds inside the set of infinitesimal action derivation.

Proposition 3.18. *Let $\Gamma = \{H_s, J_s\}_{s\in\mathbb{R}}$ be a smooth homotopy of Hamiltonians and almost complex structures defined as in Theorem 7. Fix a pair $a, b \in \mathbb{R}$. Then for every two connected components*

$$\Lambda_0 \subseteq \mathscr{C}(\mathcal{A}^{H_0}, N) \cap (\mathcal{A}^{H_0})^{-1}([a,\infty)),$$
$$\Lambda_1 \subseteq Crit(\mathcal{A}^{H_1}) \cap (\mathcal{A}^{H_1})^{-1}((-\infty, b]).$$

the corresponding $\mathscr{M}^\Gamma(\Lambda^0, \Lambda^1)$ is uniformly bounded in the $L^2 \times \mathbb{R}$ norm. Moreover, for every $\varepsilon \in (0, \varepsilon_2)$, where ε_2 is as in Proposition 3.8, then corresponding set

$$\bigcup_{u\in\mathscr{M}^\Gamma(\Lambda^0,\Lambda^1)} (u(\mathbb{R}) \cap \mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon))$$

is uniformly bounded in $W^{1,2}(S^1) \times \mathbb{R}$ norm.

Proof. Let us fix $u \in \mathscr{M}^\Gamma(\Lambda^0, \Lambda^1)$ and $s \in \mathbb{R}$. Using the sequence

$$\tau_1(s) \leq \tau_2^+(s) \leq \tau_2^-(s) \leq \tau_3^+(s) \cdots \leq s$$

associated to s as defined above, we will divide the Floer trajectory and bound it on each part differently.

If $s > 0$, then (H_s, J_s) is non-constant on $(-\infty, s]$ namely it varies on the interval $[0, \min\{1, s\}]$. We can bound there the L^2 norm of the v component using a similar approach as in Lemma 3.17 obtaining

$$\begin{aligned} \|v(\min\{1,s\}) - v(0)\|_{L^2(S^1)} &\leq \sqrt{|\min\{1,s\}| \int_0^{\min\{1,s\}} \|\partial_s v(\tau)\|^2_{L^2(S^1)} d\tau} \\ &\leq \|\nabla^{J_s} \mathcal{A}^{H_s}(u(s))\|_{L^2(S^1\times\mathbb{R})} \\ &\leq \sqrt{\mathfrak{e}} \end{aligned} \tag{3.35}$$

Recall that

$$\tau \in (\tau_1(s), \tau_2^+(s)) \bigcup_{k=2}^{K} (\tau_k^-(s), \tau_{k+1}^+(s)) \quad \Longrightarrow \quad u(\tau) \in \mathscr{N}_r^\delta.$$

That means that for any interval

$$I \subseteq (\tau_1(s), \tau_2^+(s)) \bigcup_{k=2}^{K} (\tau_k^-(s), \tau_{k+1}^+(s)) \setminus (0,1)$$

one can apply Proposition 3.16. Indeed, whenever $I \cap (0,1) = \emptyset$, then for all $\tau \in I$, H^τ is constant and equal either H^0 or H^1. Moreover, on $(-\infty, 0) \cup (1, \infty)$, $\mathcal{A}^{H_\tau}(u(\tau))$ is monotonically increasing and by Proposition 3.4

$$|\mathcal{A}^{H_\tau}(u(\tau))| \leq \mathfrak{a} \qquad \forall\, \tau \in \mathbb{R}$$

Therefore, we can apply Proposition 3.16 and combine it with (3.35) and (3.33) to estimate

$$\begin{aligned} \|v(\tau_1(s)) - v(\tau_2^+(s))\|_{L^2(S^1)} &+ \sum_{k=2}^{K} \|v(\tau_k^-(s)) - v(\tau_{k+1}^+(s))\|_{L^2(S^1)} \leq \\ &\leq \tilde{M}\big(\mathcal{A}^{H^0}(u(0)) - \mathcal{A}^{H^0}(u(\Lambda^0)) + \mathcal{A}^{H^1}(\Lambda^1) - \mathcal{A}^{H^1}(u(1))\big) \\ &+ \|v(1) - v(0)\|_{L^2(S^1)} + K2\delta \\ &\leq \sqrt{\mathfrak{e}} + 4\tilde{M}\mathfrak{a} + 4\frac{\mathfrak{e}}{\varepsilon} + 2\delta. \end{aligned} \tag{3.36}$$

Now we can finally estimate $\|v(s)\|_{L^2(S^1)}$ by using definition of $\mathcal{K}_\delta^0$ in (3.11) and combining it with the results from (3.32) and (3.36) to obtain

$$\begin{aligned} \|v(s)\|_{L^2(S^1)} &\leq \|v(\tau_1(s))\|_{L^2(S^1)} + \|v(\tau_1(s)) - v(s)\|_{L^2(S^1)} \\ &\leq v_4 + \sqrt{\mathfrak{e}} + 4\tilde{M}\mathfrak{a} + 5(\frac{\mathfrak{e}}{\varepsilon} + \delta) \end{aligned}$$

where the additional 2δ comes from the possibility that $u(s) \in \mathscr{N}_r^\delta$. We have chosen $s \in \mathbb{R}$ and $u \in \mathscr{M}^\Gamma(\Lambda^0, \Lambda^1)$ arbitrary, therefore the above inequality along with the uniform bound on η obtained in Proposition 3.4 establishes that for all $u \in \mathscr{M}^\Gamma(\Lambda^0, \Lambda^1)$

$$\|u(s)\|_{L^2(S^1)\times\mathbb{R}} \leq \mathfrak{y} + v_4 + \sqrt{\mathfrak{e}} + 4\tilde{M}\mathfrak{a} + 5(\frac{\mathfrak{e}}{\varepsilon} + \delta) \qquad \forall\ s \in \mathbb{R},$$

which proves the first claim.

Now take $u \in \mathscr{M}^\Gamma(\Lambda^0, \Lambda^1)$ and consider $s \in \mathbb{R}$, such that $u(s) \in \mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon)$. Since $\|u(s)\|_{L^2(S^1)}$ is uniformly bounded and by assumption

$$\mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon) \subseteq \mathcal{K}^1_{\delta/2} \cup \mathcal{U}^1_{\delta/2},$$

therefore using the fact that by definition of $\mathcal{K}^1_{\delta/2}$ as in (3.24) one has:

$$\sup_{x\in\mathcal{K}^1_{\delta/2}} \|x\|_{W^{1,2}\times\mathbb{R}} \leq \varepsilon_2 + v_4 + \mathfrak{y}(1 + h_1 + Mv_4)$$

we obtain a uniform bound on the $W^{1,2} \times \mathbb{R}$ norm:

$$\|u(s)\|_{W^{1,2}(S^1)\times\mathbb{R}} \leq v_4 + \max\left\{\varepsilon_2 + \mathfrak{y}(1 + h_1 + Mv_4), \sqrt{\mathfrak{e}} + 4\tilde{M}\mathfrak{a} + 6\delta + 5\frac{\mathfrak{e}}{\varepsilon}\right\}$$

Naturally, uniform bounds on the $W^{1,2}(S^1) \times \mathbb{R}$ norm induce uniform bounds on the $L^\infty \times \mathbb{R}$ norm. □

First observe that the fact that uniform $W^{1,2}(S^1) \times \mathbb{R}$ bounds on

$$\bigcup_{u\in\mathscr{M}^\Gamma(\Lambda^0,\Lambda^1)} (u(\mathbb{R}) \cap \mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon))$$

imply that Λ_1 is uniformly bounded, even if $\Lambda_1 \subseteq H_1^{-1}(0) \times \{0\}$.

Moreover, note that in the bounds obtained in the above proposition $\mathfrak{y}, \mathfrak{e}$ and ε_2 depend continuously on $\|J\|_\infty$, whereas v_4, δ and $\mathfrak{a}$ do not depend on $\|J\|_\infty$, but only of the fact that Γ satisfies (3.6) and all the Hamiltonians are in the set $H_0 + \mathscr{O}(H_0)$ and admit a common set of parameters which satisfy Definition 3.1.

3.8 Maximum principle

In this section we would like to use Aleksandrov's maximum principle to find L^∞ bounds on the Floer trajectories outside $\mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon)$, following the argument of Alberto Abbondandolo and Mathias Schwartz in [2].

Theorem 8. ***(Aleksandrov's maximum principle)***
Let Ω be a domain in $\mathbb{R}^2$ and $\rho : \Omega \to \mathbb{R}$ a C^2 function satisfying the elliptic differential inequality

$$\triangle\rho + \langle h, \nabla\rho\rangle \geq f,$$

where h and f are functions $h : \Omega \to \mathbb{R}^2$, $f : \Omega \to \mathbb{R}$. Then

$$\sup_\Omega \rho \leq \sup_{\partial\Omega} \rho + C(\|h\|_{L^2(\Omega)})\|f^-\|_{L^2(\Omega)},$$

provided provided h and the negative part f are in $L^2(\Omega)$.

In order to apply this theorem and find L^∞ bounds on the Floer trajectory, one has first to construct a plurisubharmonic function F with compact level sets, which composed with a Floer trajectory u would satisfy the elliptic differential inequality

$$\triangle(F \circ u) + \langle h, \nabla(F \circ u)\rangle \geq f \qquad \text{on} \quad \Omega \subseteq \mathbb{R} \times S^1.$$

Having such a inequality one can apply the Aleksandrov maximum principle, which gives us

$$\sup_{\Omega}(F \circ u) \leq \sup_{\partial\Omega}(F \circ u) + C(\|h\|_{L^2(\Omega)} \|f^-\|_{L^2(\Omega)}.$$

provided h and the negative part of f are in $L^2(\Omega)$.

The core of this method is to find a function, which would satisfy all the required properties. Unfortunately, the framework used in [2] cannot be translated directly to our case, since our hypersurface is not a boundary of a compact Liouville domain. However, in Proposition 3.20 , we will see present a setting, which applies to the system from Theorem 7.

3.8.1 Plurisubharmonic functions

To find a suitable function, which would fit into the framework of Aleksandrov's theorem, we first take a closer look at plurisubharmonic functions and investigate their properties.

Definition 3.19. *Let (M, ω) be a symplectic manifold and J an ω-compatible almost complex structure. Then a C^2 function $F : M \to \mathbb{R}$ is called plurisubharmonic if*

$$-dd^{\mathbb{C}} F = \omega,$$

where $d^{\mathbb{C}} = dF \circ J$.

Note that if F is plurisubharmonic then ∇F with respect to metric $\omega(\cdot, J\cdot)$ is a global Liouville vector field.

$$i_{\nabla F}\omega = \omega(\nabla F, \cdot) = \omega(J\nabla F, J\cdot) = -\omega(J\cdot, X^F) = -d^{\mathbb{C}} F.$$

Let us now focus on $(\mathbb{R}^{2n}, \omega_0)$ with the standard complex structure J_0 and analyze the quadratic, plurisubharmonic functions $F : \mathbb{R}^{2n} \to \mathbb{R}$ of the form

$$F(x) := \frac{1}{2} x^T A x - c,$$

for some symmetric matrix A and a constant c. The fact that F is plurisubharmonic forces A to satisfy

$$A - JAJ = Id.$$

Indeed if we compute

$$\begin{aligned}\nabla F(x) &= Ax,\\ \omega_0(v,w) &= -v^T J w\\ i_{\nabla F}\omega_0(v) &= -(Ax)^T Jv = -x^T AJv\\ d(i_{\nabla F}\omega_0)(v,w) &= v^T(-AJ + (AJ)^T)w\\ -AJ + (AJ)^T &= -J\\ A - JAJ &= Id.\end{aligned}$$

Note that in particular $A = \frac{1}{2}Id$ satisfies the above equation giving us a quadratic plurisubharmonic function. This gives us the canonical example of a plurisubharmonic function with compact level sets on $\mathbb{R}^{2n}$ namely the radial function

$$F(x) := \frac{1}{4}\|x\|^2.$$

The next step is to prove the elliptic inequality on the Floer trajectories. In order to achieve it, we have to investigate how the function interacts with the Hamiltonian vector field, in particular investigate the functions

$$dF(X^H) \qquad d^{\mathbb{C}}F(X^H),$$

which appear if we calculate $d^{\mathbb{C}}(F \circ u)$.As we will show in the next subsection, in order to assure that the assumptions of the Aleksandrov's maximum principle are satisfied, it is enough to require H to satisfy

$$\sup_{x\in\mathbb{R}^{2n}} \|D^3 H\|\|x\| \leq L.$$

3.8.2 Elliptic differential inequality

In this section we will prove that the radial function composed with a Floer trajectory outside the set $\mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon)$ satisfies the assumptions of the Aleksandrov's maximum principle, provided u is globally bounded in the $L^2 \times \mathbb{R}$ norm and we choose the Hamiltonian function to satisfy

$$\sup_{x\in\mathbb{R}^{2n}} \|D^3 H\|\|x\| \leq L.$$

This statement is proven in the following Proposition.

Proposition 3.20. *Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a Hamiltonian and $u : \mathbb{R}\times S^1 \to \mathbb{R}^{2n+1}$, $u(s) = (v(s), \eta(s))$ be a solution to the Floer equation corresponding to the constant almost complex structure J_0. Then for the radial, plurisubharmonic function*

$$F(x) := \frac{1}{4}\|x\|^2$$

there exists a function $f : \mathbb{R} \times S^1 \to \mathbb{R}$*, such that*

$$\triangle(F \circ v) \geq f(s,t).$$

Moreover, if we assume the following

1. *H satisfies (H2)*
2. *there exist constants* $\mathfrak{e}, \mathbf{v}, \mathfrak{y} > 0$*, such that*

$$\|\nabla \mathcal{A}^H(u)\|^2_{L^2(\mathbb{R}\times S^1)} \leq \mathfrak{e}$$
$$\|v(s)\|_{L^2(S^1)} \leq \mathbf{v}, \qquad |\eta(s)| \leq \mathfrak{y}, \qquad \forall s \in \mathbb{R}$$

3. *for a constant* $\varepsilon > 0$*,* $\Omega \subseteq \mathbb{R} \times S^1$ *is a connected, open subset, such that if we define*

$$s_0 := \inf_{(s,t)\in\Omega} s \qquad s_1 := \sup_{(s,t)\in\Omega} s$$

then for all $s \in (s_0, s_1)$ *one has* $\|\nabla \mathcal{A}^H(u(s))\| > \varepsilon$*.*

Then

$$\|f\|_{L^2(\Omega)} \leq C(\mathfrak{e}, \mathfrak{y}, \mathbf{v}, \varepsilon) < +\infty$$

where the constant C depends only on the parameters of the Hamiltonian and the constants $\varepsilon, \mathbf{v}, \mathfrak{y}, \mathfrak{e}$ *in a continuous way.*

Proof. From the Floer equations we have

$$\begin{aligned} \partial_s v &= -J(\partial_t v - \eta X^H) \\ \partial_t v &= J\partial_s v + \eta X^H \end{aligned} \tag{3.37}$$

If we plug the above into $d^{\mathbb{C}}(F \circ v)$ we obtain

$$\begin{aligned} d^{\mathbb{C}}(F \circ v) &= dF(\partial_t v)ds - dF(\partial_s v)dt \\ &= dF(J\partial_s v + \eta X^H))ds + dF(J(\partial_t v - \eta X^H))dt \\ &= v^*(d^{\mathbb{C}} F) + \eta(dF(X^H)ds - d^{\mathbb{C}} F(X^H)dt) \\ dd^{\mathbb{C}}(F \circ v) &= v^*(dd^{\mathbb{C}} F) - (\eta d(dF(X^H)(\partial_t v) + \partial_s \eta \; d^{\mathbb{C}} F(X^H) \\ &\qquad + \eta d(d^{\mathbb{C}} F(X^H))(\partial_s v))ds \wedge dt \end{aligned}$$

Let us consider the two parts of the above expression separately. Using the ω_0 compatibility of J and plurisubharmonicity of F, we can compute

$$\begin{aligned} v^*(-dd^{\mathbb{C}} F) &= \omega_0(\partial_s v, \partial_t v) ds \wedge dt \\ &= \omega_0(\partial_s v, J\partial_s v + \eta X^H) ds \wedge dt \\ &= (\|\partial_s v\|^2 + \eta dH(\partial_s v)) ds \wedge dt. \end{aligned}$$

Let us now combine the above results together with the fact that

$$-dd^{\mathbb{C}}(F\circ v) = \triangle(F\circ v)ds\wedge dt,$$

and compute

$$\begin{aligned}
\triangle(F\circ v) &= \omega_0(\partial_s v,\partial_t v) + \eta d(dF(X^H)(\partial_t v) + \partial_s\eta\ d^{\mathbb{C}}F(X^H) + \eta d(d^{\mathbb{C}}F(X^H))(\partial_s v)\\
&= \|\partial_s v\|^2 + \eta(dH(\partial_s v) + d(d^{\mathbb{C}}F(X^H))(\partial_s v) + d(dF(X^H))(J\partial_s v + \eta X^H)\\
&\qquad\qquad + \partial_s\eta\ d^{\mathbb{C}}F(X^H)\\
&= \|\partial_s v + \eta(\nabla H + \nabla(d^{\mathbb{C}}F(X^H)) - J\nabla(dF(X^H)))\|^2 + \eta^2 d(dF(X^H))(X^H)\\
&\qquad - \eta^2\|\nabla H + \nabla(d^{\mathbb{C}}F(X^H)) - J\nabla(dF(X^H))\|^2 + \partial_s\eta\ d^{\mathbb{C}}F(X^H)\\
&\geq \eta^2(d(dF(X^H))(X^H) - \|\nabla H\|^2 - \|\nabla(d^{\mathbb{C}}F(X^H))\|^2 - \|\nabla(dF(X^H))\|^2)\\
&\qquad\qquad + \partial_s\eta\ d^{\mathbb{C}}F(X^H).
\end{aligned}$$

Denote

$$\begin{aligned}
f(s,t) &:= \eta^2(s)(d_{v(s,t)}(dF(X^H))(X^H) - \|\nabla H(v(s,t))\|^2 - \|\nabla(d^{\mathbb{C}}F(X^H))(v(s,t))\|^2\\
&\quad - \|\nabla(dF(X^H))(v(s,t)\|^2) + \partial_s\eta(s)\ d^{\mathbb{C}}_{v(s,t)}F(X^H).
\end{aligned}\tag{3.38}$$

We will prove that $f\in L^2(\Omega)$ by first proving that $f\in W^{1,1}(\Omega)$ and then using the Sobolev embedding

$$W^{1,1}(\Omega)\hookrightarrow L^2(\Omega).$$

Let us now analyze the assumptions on H and how do they imply the boundedness of f in $W^{1,1}$. Note that the assumption

$$\sup_{x\in\mathbb{R}^{2n}}\|D^3H\|\|x\|\leq L,$$

implies quadratic behavior of the Hamiltonian, i.e. for every $x\in\mathbb{R}^{2n}$

$$\|Hess_x H\|\leq M, \tag{3.39}$$

$$\|\nabla H(x)\|\leq h_1 + M\|x\|, \tag{3.40}$$

$$|H(x)|\leq h_0 + h_1\|x\| + \frac{1}{2}M\|x\|^2, \tag{3.41}$$

$$\text{where}\qquad M := \|Hess_0 H\| + L,\qquad h_0 := |H(0)|\qquad h_1 := \|\nabla H(0)\|.$$

Denote now a part of f by

$$f_1(x) := d_x(dF(X^H))(X^H) - \|\nabla H(x)\|^2 - \|\nabla(d^{\mathbb{C}}_x F(X^H))\|^2 - \|\nabla(dF(X^H))(x)\|^2. \tag{3.42}$$

Then the quadratic behavior of H forces $|f_1|$ to be maximally of order 2 in $\|x\|$ and $\|\nabla f_1(x)\|$ to be linear in $\|x\|$, meaning

$$|f_1(x)|\leq(2M\|x\| + \frac{2}{3}h_1)^2, \tag{3.43}$$

$$\|\nabla f_1(x)\|\leq h_2\|x\| + h_3, \tag{3.44}$$

$$\text{where}\quad h_2 := M(8M + \frac{5}{2}L),\quad h_3 := \frac{1}{2}h_1(7M + 3L).$$

The explicit calculations of the above statements are carried out in Appendix A, Lemma 4.15.

Now we can express f and its derivatives in terms of f_1.

$$\begin{aligned}
f(s,t) &= \eta^2(s) f_1(v(s,t)) + \partial_s\eta(s) d^{\mathbb{C}} F_{v(s,t)}(X^H), && (3.45)\\
\partial_t f(s,t) &= \eta^2 df_1(\partial_t v) + \partial_s \eta d(d^{\mathbb{C}} F_{v(s,t)}(X^H))(\partial_t v)\\
&= \langle \eta^2 \nabla f_1(v) + \partial_s\eta \nabla(d^{\mathbb{C}} F_{v(s,t)}(X^H)), \partial_t v\rangle, && (3.46)\\
\partial_s f(s,t) &= \eta^2 df_1(\partial_s v) + \partial_s\eta\ d(d^{\mathbb{C}} F_{v(s,t)}(X^H))(\partial_s v)\\
&\quad + 2\eta\partial_s\eta f_1(v(s,t)) - d^{\mathbb{C}} F_{v(s,t)}(X^H)\int_0^1 dH(\partial_s v)dt. && (3.47)
\end{aligned}$$

Observe that the functions f_1 and $d^{\mathbb{C}} F_x(X^H)$ are at most quadratic in $\|x\|$:

$$\begin{aligned}
|f_1(x)| &\le (2M\|x\| + \frac{2}{3}h_1)^2,\\
|d^{\mathbb{C}} F_x(X^H)| &\le \frac{1}{2}\|x\|(M\|x\| + h_1),
\end{aligned}$$

whereas their derivatives are linear in $\|x\|$

$$\begin{aligned}
\|\nabla f_1(x)\| &\le h_2\|x\| + h_3,\\
\|\nabla(d^{\mathbb{C}} F(X^H))\| &\le M\|x\| + \frac{1}{2}h_1.
\end{aligned}$$

This, together with the global bound in the $L^2 \times \mathbb{R}$ norm on $u(s)$, the finiteness of the energy

$$\|\nabla \mathcal{A}^H(u)\|^2_{L^2(\mathbb{R}\times S^1)} = \int_{\mathbb{R}} \Big(|\partial_s\eta(s)|^2 + \int_0^1 \|\partial_s v(s,t)\|^2 dt\Big) ds \le \mathfrak{e},$$

and the bound on the length of the interval estimated in Lemma 3.17, equation (3.34)

$$u(s) \notin \mathcal{B}^J(\mathfrak{a}, \mathfrak{y}, \varepsilon) \quad \forall\ s \in [s_0, s_1], \qquad \Rightarrow \qquad |s_1 - s_0| \le \frac{\mathfrak{e}}{\varepsilon^2},$$

enables us to show that the $W^{1,1}$ norm of f on Ω is bounded by a constant, which depends only on the parameters of the Hamiltonian, the chosen value of ε and the constants $\mathbf{v}, \mathfrak{y}, \mathfrak{e}$.

The explicit estimates on $\|f\|_{L^1(\Omega)}$ and $\|\nabla f\|_{L^1(\Omega)}$ are calculated in Appendix A, Lemma 4.16. $\square$

3.9 Invariance of RFH under perturbation

The aim of this section is to show that in the class of tentacular Hamiltonians the Rabinowitz Floer homology is independent of the choice of almost complex structure and invariant under small compactly supported perturbations of Hamiltonians. Those two results are presented in Lemma 3.21 and Lemma 3.22 respectively. The techniques used to prove the results have been already discussed in a general setting in Section 2.6. Unfortunately, to prove functoriality, i.e. that a homorphism of a concatenation of homotopies is a composition of homomorphisms as in Proposition 2.19 one has to use homotopies, which support is not contained in the interval $[0, 1]$ as we assumed in the proof of Theorem 7. However, we claim that uniform bounds can be proven using the same techniques as presented in Theorem 7 as long as the measure of the support is uniformly bounded.

Lemma 3.21. *Fix a Hamiltonian* $\mathsf{H} \in \mathscr{H}^{reg}$*, i.e.* $\mathsf{H} \in \mathscr{H}$ *and* H *satisfying property (MB). Let* $\mathscr{J}^{reg}(\mathsf{H})$ *be the set of regular 2-parameter families of almost complex structures in the sense of Theorem 2. Then for every two* $\mathsf{J}_1, \mathsf{J}_2 \in \mathscr{J}^{reg}(H)$ *the corresponding Rabinowitz Floer homologies are isomorphic, i.e.*

$$RFH(\mathsf{H}, \mathsf{J}_1) \cong RFH(\mathsf{H}, \mathsf{J}_2).$$

Proof. By assumption $\mathsf{H} \in \mathscr{H}$, hence by Property (H4) and Corollary 2.28 it satisfies Property (PO+). In particular there exists an open, precompact subset $\mathcal{V} \subseteq \mathbb{R}^{2n}$ and a constant $\mathfrak{y} > 0$, such that

$$\forall\ (v, \eta) \in (Crit(\mathcal{A}^{\mathsf{H}}) \setminus (\mathcal{A}^{\mathsf{H}})^{-1}(0)), \qquad v(S^1) \subseteq \mathcal{V} \quad \& \quad |\eta| > \mathfrak{y}$$

and

$$\mathsf{J}_1, \mathsf{J}_2 \in \mathscr{J}\Big(\mathbb{R}^{2n}, \omega_0, \mathcal{V} \times \big((-\infty, -\mathfrak{y}) \cup (\mathfrak{y}, +\infty)\big)\Big).$$

The above set of almost complex structures is simply connected, therefore there exists a smooth homotopy $\{J_s\}_{s\in\mathbb{R}}$, such that

$$\begin{aligned} &\forall\ s \in \mathbb{R} \quad J_s \in \mathscr{J}\Big(\mathbb{R}^{2n}, \omega_0, \mathcal{V} \times \big((-\infty, -\mathfrak{y}) \cup (\mathfrak{y}, +\infty)\big)\Big), \\ &s \leq 0 \quad J_s := \mathsf{J}_1 \qquad \& \qquad s \geq 1 \quad J_s := \mathsf{J}_2. \end{aligned}$$

Fix a a compact set $K \subseteq \mathbb{R}^{2n},\ K \neq \emptyset$. By Lemma 3.2 to the pair H and K one can associate a constant $\theta > 0$, such that set

$$\mathscr{O}(\mathsf{H}) := \{h \in C_0^\infty(K) \mid \|h\|_{C^3(K)} < \theta\}$$

is not only open in $C_0^\infty(K)$, but also

$$\mathsf{H} + \mathscr{O}(\mathsf{H}) \subseteq \mathscr{H}.$$

Moreover, by Lemma 3.2 we can choose a uniform set of parameters for the whole set $\mathsf{H} + \mathscr{O}(\mathsf{H})$, hence the constants $\tilde{c}$ and ε_0 from Lemma 3.5 can be chosen uniformly for the whole set $\mathsf{H} + \mathscr{O}(\mathsf{H})$. Now consider the set

$$\mathscr{O}(\mathsf{J}_1, \mathsf{J}_2) := \left\{ \quad h \in C_0^\infty([0,1] \times K) \;\middle|\; \begin{array}{c} \|\partial_s h\|_\infty < \frac{1}{32}(\tilde{c} + \frac{1}{\varepsilon_0}\{\|J\|_\infty^3, 1\})^{-1} \\ \forall\ s \in [0,1] \quad \|h_s\|_{C^3(K)} < \frac{1}{2}\theta \end{array} \right\} \tag{3.48}$$

where θ is as in definition of $\mathscr{O}(\mathsf{H})$. Then $\mathscr{O}(\mathsf{J}_1, \mathsf{J}_2)$ is a neighborhood of 0 in $C_0^\infty([0,1]\times K)$. Moreover, for every $h \in \mathscr{O}(\mathsf{J}_1, \mathsf{J}_2)$ the corresponding homotopy $\Gamma(h)$ defined

$$\Gamma(h) := \{(\mathsf{H} + h_s, J_s)\}_{s\in\mathbb{R}}$$

satisfies inequality (3.6) and the Hamiltonian part of the homotopy lies in $\mathsf{H} + \mathscr{O}(\mathsf{H})$. In other words, for every $h \in \mathscr{O}(\mathsf{J}_1, \mathsf{J}_2)$ the corresponding homotopy $\Gamma(h)$ satisfies the assumptions of Theorem 7 and as a result one obtains uniform $L^\infty \times \mathbb{R}$ bounds on the corresponding moduli spaces and uniform bounds on the energy as shown in Proposition 3.4. Moreover, using the linearity condition between the action and the η, as stated in Lemma 3.5, together with inequality (3.6), we can directly apply Corollary 3.8 from [12] to show that every $\Gamma(h)$ satisfies the Novikov conditions. In consequence, the homotopy $\{\mathsf{H}, J_s\}_{s\in\mathbb{R}}$ and the set $\mathscr{O}(\mathsf{J}_1, \mathsf{J}_2)$ satisfy the assumptions of Lemma 2.15 and we can deduce that for a generic choice of $h \in \mathscr{O}(\mathsf{J}_1, \mathsf{J}_2)$ there exists a homomorphism $\Psi^{\Gamma(h)}$

$$\Psi^{\Gamma(h)} : RFH(\mathsf{H}, \mathsf{J}_2) \to RFH(\mathsf{H}, \mathsf{J}_1).$$

Note that for every $h \in \mathscr{O}(\mathsf{J}_1, \mathsf{J}_2)$ the corresponding homotopy $\Gamma^{-1}(h)$, defined by

$$\Gamma^{-1}(h) := \{(\mathsf{H} + h_{1-s}, J_{1-s})\}_{s\in\mathbb{R}},$$

also satisfies inequality (3.6) and its Hamiltonian part lies in $\mathsf{H} + \mathscr{O}(\mathsf{H})$. By the same argument as the one above, we can conclude that for a generic choice of $h \in \mathscr{O}(\mathsf{J}_1, \mathsf{J}_2)$ all four homotopies, namely $\Gamma(h)$, its inverse $\Gamma^1(h)$ and their concatenations

$$\begin{aligned}
\Gamma(h)\#\Gamma^{-1}(h) &:= \{(\mathsf{H} + \mathsf{h}_{1-|2s-1|}, J_{1-|2s-1|})\}_{s\in\mathbb{R}},\\
\Gamma^{-1}(h)\#\Gamma(h) &:= \{(\mathsf{H} + \mathsf{h}_{|2s-1|}, J_{|2s-1|}\}_{s\in\mathbb{R}}
\end{aligned}$$

are regular, i.e. the four homomorphisms in the figure below are well defined.

$$\begin{tikzcd}
RFH(\mathsf{H}, \mathsf{J}_1) \arrow[loop left, "\Psi^{\Gamma(h)\#\Gamma^{-1}(h)}"] \arrow[rr, bend left, "\Psi^{\Gamma^{-1}(h)}"] & & RFH(\mathsf{H}, \mathsf{J}_2) \arrow[loop right, "\Psi^{\Gamma^{-1}(h)\#\Gamma(h)}"] \arrow[ll, bend left, "\Psi^{\Gamma(h)}"]
\end{tikzcd}$$

Now fix $\mathsf{h} \in \mathscr{O}(\mathsf{J}_1, \mathsf{J}_2)$ and construct two corresponding homotopies of homotopies $\widetilde{\Gamma} = \{\widetilde{\Gamma}^\lambda\}_{\lambda\in[0,1]}$ and $\widehat{\Gamma} = \{\widehat{\Gamma}^\lambda\}_{\lambda\in[0,1]}$ by defining for every $\lambda \in [0,1]$

$$\begin{aligned}
\widetilde{\Gamma}^\lambda &:= \{(\mathsf{H} + \mathsf{h}_{\lambda(1-|2s-1|)}, J_{\lambda(1-|2s-1|)})\}_{s\in\mathbb{R}},\\
\widehat{\Gamma}^\lambda &:= \{(\mathsf{H} + \mathsf{h}_{1-\lambda(1-|2s-1|)}, J_{1-\lambda(1-|2s-1|)})\}_{s\in\mathbb{R}}.
\end{aligned}$$

Then $\widetilde{\Gamma}$ and $\widehat{\Gamma}$ satisfy

$$\begin{aligned}
\widetilde{\Gamma}^0 &= \{(\mathsf{H}, \mathsf{J}_1)\}_{s\in\mathbb{R}} = Id_{(\mathsf{H},\mathsf{J}_1)}, &\qquad \widetilde{\Gamma}^1 &= \Gamma(\mathsf{h})\#\Gamma(\mathsf{h})^{-1},\\
\widehat{\Gamma}^0 &= \{(\mathsf{H}, \mathsf{J}_2)\}_{s\in\mathbb{R}} = Id_{(\mathsf{H},\mathsf{J}_2)}, &\qquad \widehat{\Gamma}^1 &= \Gamma(\mathsf{h})^{-1}\#\Gamma(\mathsf{h}).
\end{aligned}$$

Moreover, for all $\lambda \in [0,1]$ the corresponding $\widetilde{\Gamma}^\lambda$ and $\widehat{\Gamma}^\lambda$ start and end in $(\mathsf{H}, \mathsf{J}_1)$ and $(\mathsf{H}, \mathsf{J}_2)$ respectively, i.e. for all $s \in (-\infty, 0] \cup [1, +\infty)$

$$(\mathsf{H} + \mathsf{h}_{\lambda(1-|2s-1|)}, J_{\lambda(1-|2s-1|}) = (\mathsf{H}, \mathsf{J}_1),$$
$$(\mathsf{H} + \mathsf{h}_{1-\lambda(1-|2s-1|)}, J_{1-\lambda(1-|2s-1|}) = (\mathsf{H}, \mathsf{J}_2).$$

Now we will construct an open neighborhood of 0 in $C_0^\infty([0,1]^2 \times K)$

$$\mathscr{O}(\Gamma(\mathsf{h})) := \left\{ \quad h \in C_0^\infty([0,1]^2 \times K) \;\middle|\; \begin{array}{c} \|\partial_s h\|_\infty < \frac{1}{16}(\tilde{c} + \frac{1}{\varepsilon_0} \max\{\|J\|_\infty^3, 1\})^{-1} \\ \forall\ (\lambda, s) \in [0,1]^2 \quad \|h_s^\lambda\|_{C^3(K)} < \frac{1}{2}\theta \end{array} \right\} \tag{3.49}$$

If we take $h \in \mathscr{O}(\Gamma(\mathsf{h}))$ and denote

$$\widetilde{\Gamma}^\lambda(h) := \{(\mathsf{H} + \mathsf{h}_{\lambda(1-|2s-1|)} + h_s^\lambda, J_{\lambda(1-|2s-1|})\}_{s\in\mathbb{R}},$$
$$\widehat{\Gamma}^\lambda(h) := \{(\mathsf{H} + \mathsf{h}_{\lambda|2s-1|} + h_s^\lambda, J_{\lambda|2s-1|}\}_{s\in\mathbb{R}},$$

then $\{\widetilde{\Gamma}^\lambda(h)\}_{\lambda\in[0,1]}$ is a homotopy of homotopies between $Id_{(\mathsf{H},\mathsf{J}_1)}$ and $\Gamma(\mathsf{h})\#\Gamma(\mathsf{h})^{-1}$, whereas $\{\widehat{\Gamma}^\lambda(h)\}_{\lambda\in[0,1]}$ is a homotopy of homotopies between $Id_{(\mathsf{H},\mathsf{J}_2)}$ and $\Gamma^{-1}(\mathsf{h})\#\Gamma(\mathsf{h})$. In particular, $\{\widetilde{\Gamma}^\lambda(h)\}_{\lambda\in[0,1]}$ and $\{\widehat{\Gamma}^\lambda(h)\}_{\lambda\in[0,1]}$ are compact perturbations of $\{\widetilde{\Gamma}^\lambda\}_{\lambda\in[0,1]}$ and $\{\widehat{\Gamma}^\lambda\}_{\lambda\in[0,1]}$.

Now we will show that for every $\lambda \in [0,1]$ and $h \in \mathscr{O}(\Gamma(\mathsf{h}))$ the corresponding $\widetilde{\Gamma}^\lambda(h)$ satisfies inequality (3.6) and the Hamiltonian part is in $\mathsf{H} + \mathscr{O}(\mathsf{H})$. Indeed, for every $(s,\lambda) \in [0,1]^2$ using (3.48) and (3.49) we can calculate

$$\|\mathsf{h}_{\lambda(1-|2s-1|)} + h_s^\lambda\|_{C^3(K)} \leq \max_{s\in[0,1]} \|\mathsf{h}_s\|_{C^3(K)} + \max_{(s,\lambda)\in[0,1]^2} \|h_s^\lambda\|_{C^3(K)} < \theta,$$

which proves the second claim, i.e. that for every $(s,\lambda) \in [0,1]^2$

$$\mathsf{h}_{\lambda(1-|2s-1|)} + h_s^\lambda \in \mathscr{O}(\mathsf{H}).$$

Observe that for every $\lambda \in [0,1]$ we have

$$\|J_{\lambda(1-|2s-1|)}\|_\infty \leq \|J\|_\infty. \tag{3.50}$$

Therefore, by using again (3.48) and (3.49) and calculating

$$\begin{aligned}
\|\partial_s(\mathsf{h}_{\lambda(1-|2s-1|)} + h_s^\lambda)\|_\infty &\leq 2\lambda\|\partial_s \mathsf{h}\|_\infty + \|\partial_s h\|_\infty \\
&\leq 2\|\partial_s \mathsf{h}\|_\infty + \|\partial_s h\|_\infty \\
&\leq \big(2 \cdot \frac{1}{32} + \frac{1}{16}\big)(\tilde{c} + \frac{1}{\varepsilon_0} \max\{\|J\|_\infty^3, 1\})^{-1} \\
&= \frac{1}{8}(\tilde{c} + \frac{1}{\varepsilon_0} \max\{\|J\|_\infty^3, 1\})^{-1} \\
&\leq \frac{1}{8}(\tilde{c} + \frac{1}{\varepsilon_0} \max\{\|J_{\lambda(1-|2s-1|)}\|_\infty^3, 1\})^{-1}
\end{aligned}$$

we can see that for every $\lambda \in [0,1]$ and $h \in \mathscr{O}(\Gamma(\mathsf{h}))$ the corresponding $\widetilde{\Gamma}^\lambda(h)$ satisfies inequality (3.6).

Now we would like to show that for every $h \in \mathscr{O}(\Gamma(\mathsf{h}))$ the family $\{\widetilde{\Gamma}^\lambda(h)\}_{\lambda\in[0,1]}$ satisfies assumptions of Lemma 2.17.

First note that for all $\lambda \in [0,1]$ the corresponding $\widetilde{\Gamma}^\lambda(h)$ satisfies inequality (3.6) and the Hamiltonian part is in $\mathsf{H}+\mathscr{O}(\mathsf{H})$, therefore one can use the linearity condition between the action and the η as stated in Lemma 3.5 together with inequality (3.6) and direclty apply Corollary 3.8 from [12] to obtain that for every pair $(a,b) \in Crit(\mathcal{A}^{\mathsf{H}})^2$

$$A(\widetilde{\Gamma}^\lambda(h),a,b) \geq \min\{2a,-1\},$$
$$B(\widetilde{\Gamma}^\lambda(h),a,b) \leq \max\{2b,1\}.$$

for $A(\widetilde{\Gamma}^\lambda(h),a,b)$ and $B(\widetilde{\Gamma}^\lambda(h),a,b)$ defined as in (2.23) and (2.24) respectively. That means that the family $\{\widetilde{\Gamma}^\lambda(h)\}_{\lambda\in[0,1]}$ satisfies Novikov finiteness condition uniformly.

Moreover, for all $\lambda \in [0,1]$ the corresponding $\widetilde{\Gamma}^\lambda(h)$ satisfies inequality (3.6) and the Hamiltonian part is in $\mathsf{H}+\mathscr{O}(\mathsf{H})$, therefore by Proposition 3.4 for a fixed $\lambda \in [0,1]$ one obtains uniform bounds on the energy of the perturbed Floer trajectories corresponding to $\widetilde{\Gamma}^\lambda(h)$ in a given action window. In particular, as we can see in (3.19) the bounds depend on the action window and continuously on the maximum of the homotopy of the almost complex structures $\|J\|_\infty$. But for all $\lambda \in [0,1]$ we have a uniform bound on the maximum of the homotopy of the almost complex structures as shown in (3.50). Therefore, for a fixed action window there exists a uniform bound on the energy of the perturbed Floer trajectories for the whole family $\{\widetilde{\Gamma}^\lambda(h)\}_{\lambda\in[0,1]}$.

Finally, observe that in the proof of Theorem 7 in the bounds obtained in Proposition 3.18 and Proposition 3.20 depend continuously on the parameters $N,\mathfrak{y},\mathfrak{e},\mathfrak{a},\delta,\varepsilon_2$ and v_4, where N is any fixed compact subset of $\mathsf{H}^{-1}(0)$ and the rest of the parameters is defined in (3.18), (3.19), (3.20), (3.9), Proposition 3.8 and (3.10) respectively. Moreover, $\mathfrak{y},\mathfrak{e}$ and ε_2 depend continuously on $\|J\|_\infty$, whereas v_4,δ and $\mathfrak{a}$ do not depend on $\|J\|_\infty$, but only on Γ satisfying (3.6) and all the Hamiltonians being in the set $\mathsf{H}+\mathscr{O}(\mathsf{H})$ and admitting a common set of parameters which satisfy Definition 3.1. This way, having a uniform bound on the maximum of the homotopy of the almost complex structures as shown in (3.50), we obtain bounds in the $L^\infty\times\mathbb{R}$ norm on the moduli spaces of the perturbed Floer trajectories for the whole family $\{\widetilde{\Gamma}^\lambda(h)\}_{\lambda\in[0,1]}$, i.e. independent of $\lambda\in[0,1]$.

Having shown uniform bounds for the family $\{\widetilde{\Gamma}^\lambda(h)\}_{\lambda\in[0,1]}$ we can now apply Lemma 2.17 and conclude that the homomorphisms $\Psi^{\Gamma(\mathsf{h})\#\Gamma(\mathsf{h})^{-1}}$ and $\Psi^{Id_{(\mathsf{H},\mathsf{J}_1)}}$ are chain homotopic, so

$$\Psi^{\Gamma(\mathsf{h})\#\Gamma(\mathsf{h})^{-1}} = \Psi^{Id_{(\mathsf{H},\mathsf{J}_1)}} = Id. \tag{3.51}$$

In a similar way we can show uniform bounds for the family $\{\widehat{\Gamma}^\lambda(h)\}_{\lambda\in[0,1]}$ and conclude that

$$\Psi^{\Gamma^{-1}(\mathsf{h})\#\Gamma(\mathsf{h})} = \Psi^{Id_{(\mathsf{H},\mathsf{J}_2)}} = Id. \tag{3.52}$$

Now we would like to show that

$$\Psi^{\Gamma(\mathsf{h})\#\Gamma^{-1}(\mathsf{h})} = \Psi^{\Gamma(\mathsf{h})} \circ \Psi^{\Gamma^{-1}(\mathsf{h})},$$
$$\Psi^{\Gamma^{-1}(\mathsf{h})\#\Gamma(\mathsf{h})} = \Psi^{\Gamma^{-1}(\mathsf{h})} \circ \Psi^{\Gamma(\mathsf{h})}.$$

by applying Proposition 2.19.

For every $R \geq 0$ denote

$$\overline{\Gamma}^R := \{(\mathsf{H} + \mathsf{h}_{(1+e^{-R})(R+\frac{1}{2}-|s-\frac{1}{2}|)}, J_{(1+e^{-R})(R+\frac{1}{2}-|s-\frac{1}{2}|)})\}_{s\in\mathbb{R}}$$
$$\widetilde{\Gamma}^R := \{(\mathsf{H} + \mathsf{h}_{(1+e^{-R})(|s-\frac{1}{2}|-R)}, J_{(1+e^{-R})|s-\frac{1}{2}|-R)}\}_{s\in\mathbb{R}}.$$

Naturally

$$\overline{\Gamma}^0 = \Gamma(\mathsf{h})\#\Gamma^{-1}(\mathsf{h}), \qquad \text{and} \qquad \widetilde{\Gamma}^0 = \Gamma^{-1}(\mathsf{h})\#\Gamma(\mathsf{h}).$$

Unfortunately, for $R > 0$ the homotopies $\overline{\Gamma}^R$ and $\widetilde{\Gamma}^R$ are not in the same class of homotopies as in Theorem 7, therefore we cannot apply it directly. However, there are constant outside of a compact set and the measure of the set where $\overline{\Gamma}^R$ and $\widetilde{\Gamma}^R$ are non-constant is uniformly bounded. Indeed, observe that for every $R \geq 0$

$$\begin{aligned} supp(\mathsf{h}_{(1+e^{-R})(R+\frac{1}{2}-|s-\frac{1}{2}|)}) &= supp(\partial_s(J_{(1+e^{-R})(R+\frac{1}{2}-|s-\frac{1}{2}|)})) \\ &\subseteq [-R, -R + (1 + e^{-R})^{-1}] \\ &\qquad \cup [R + 1 - (1 + e^{-R})^{-1}, R + 1] \\ &\subseteq [-R, -R + 1] \cup [R, R + 1] \end{aligned}$$

which means that the corresponding homotopy $\overline{\Gamma}^R$ is constant outside of $[-R, -R + 1] \cup [R, R + 1]$ of measure 2. Analogically,

$$\begin{aligned} supp(\mathsf{h}_{(1+e^{-R})(|s-\frac{1}{2}|-R)}) &= supp(\partial_s(J_{(1+e^{-R})|s-\frac{1}{2}|-R)})) \\ &\subseteq [-R - (1 + e^{-R})^{-1} + \frac{1}{2}, -R + \frac{1}{2}] \\ &\qquad \cup [R + \frac{1}{2}, R + \frac{1}{2} + (1 + e^{-R})^{-1}] \\ &\subseteq [-R - \frac{1}{2}, -R + \frac{1}{2}] \cup [R + \frac{1}{2}, R + \frac{3}{2}] \end{aligned}$$

so that the corresponding homotopy $\overline{\Gamma}^R$ is constant outside of $[-R - \frac{1}{2}, -R + \frac{1}{2}] \cup [R + \frac{1}{2}, R + \frac{3}{2}]$ of measure 2.

We claim that the fact that families $\{\overline{\Gamma}^R\}_{R\geq 0}$ and $\{\widetilde{\Gamma}^R\}_{R\geq 0}$ satisfy the following:

(i) the Hamiltonian part is in the set $\mathsf{H} + \mathscr{O}(\mathsf{H})$

(ii) the almost complex structures are in the set

$$\mathscr{J}\Big(\mathbb{R}^{2n}, \omega_0, \mathcal{V} \times \big((-\infty, \eta) \cup (\eta, +\infty)\big)\Big),$$

and the maximum over all almost complex structures over the whole family of homotopies is uniformly bounded

(iii) the measure of the set over which the homotopies are non-constant in s is uniformly bounded

(iv) every homology satisfies inequality (3.6) or a possibly stricter similar one

is sufficient to show that the moduli spaces of $\{\overline{\Gamma}^R\}_{R\geq 0}$ and $\{\widetilde{\Gamma}^R\}_{R\geq 0}$ are uniformly bounded in R in a similar proof as for Theorem 7. Having obtained uniform bounds we can apply Proposition 2.19 and combine it with (3.51) and (3.52) to obtain

$$\begin{aligned}
\Psi^{\Gamma(\mathsf{h})} \circ \Psi^{\Gamma^{-1}(\mathsf{h})} &= \Psi^{\Gamma(\mathsf{h})\#\Gamma^{-1}(\mathsf{h})} = \Psi^{Id_{(\mathsf{H},\mathsf{J}_1)}} = Id,\\
\Psi^{\Gamma^{-1}(\mathsf{h})} \circ \Psi^{\Gamma(\mathsf{h})} &= \Psi^{\Gamma^{-1}(\mathsf{h})\#\Gamma(\mathsf{h})} = \Psi^{Id_{(\mathsf{H},\mathsf{J}_2)}} = Id,
\end{aligned}$$

which concludes the proof that

$$\Psi^{\Gamma(\mathsf{h})} : RFH(\mathsf{H},\mathsf{J}_2) \to RFH(\mathsf{H},\mathsf{J}_1)$$

is an isomorphism. □

Having proven that the Rabinowitz Floer homology for tentacular Hamiltonians is independent of the choice of the almost complex structure, now we would like to show that it persists under small, compactly supported perturbations of Hamiltonians. The proof of the following is similar to the proof of the lemma above, as it uses the same techniques as presented in Section 2.6.

Lemma 3.22. *Fix a tentacular Hamiltonian $\mathsf{H} \in \mathscr{H}$ and compact set $K \subseteq \mathbb{R}^{2n}$, $K \neq \emptyset$. Then there exists $\widetilde{\mathscr{O}}(\mathsf{H})$ an open neighborhood of 0 in $C_0^\infty(K)$, such that for every pair*

$$\mathsf{h}_1, \mathsf{h}_2 \in \{h \in \widetilde{\mathscr{O}}(\mathsf{H}) \mid \mathsf{H} + h \in \mathscr{H}^{reg}\}$$

the corresponding Rabinowitz Floer homologies are isomorphic, i.e.

$$RFH(\mathsf{H}+\mathsf{h}_1) \cong RFH(\mathsf{H}+\mathsf{h}_2).$$

Proof. By Lemma 3.2, to the pair H and K one can associate a constant $\theta > 0$, such that set

$$\mathscr{O}(\mathsf{H}) := \{h \in C_0^\infty(K) \mid \|h\|_{C^3(K)} < \theta\}$$

is not only open in $C_0^\infty(K)$, but also

$$\mathsf{H} + \mathscr{O}(\mathsf{H}) := \{\mathsf{H} + h \mid h \in \mathscr{O}(\mathsf{H})\} \subseteq \mathscr{H}.$$

Moreover, by Lemma 3.2 we can choose a uniform set of parameters for the whole set $\mathsf{H} + \mathscr{O}(\mathsf{H})$, hence the constants $\tilde{c}$ and ε_0 from Lemma 3.5 can be chosen uniformly for the whole set $\mathsf{H} + \mathscr{O}(\mathsf{H})$. Denote

$$\widetilde{\mathscr{O}}(\mathsf{H}) := \left\{ \quad h \in C_0^\infty(K) \;\middle|\; \begin{array}{c} \|h\|_\infty < 2^{-7}\big(\tilde{c} + \frac{1}{\varepsilon_0}\big)^{-1} \\ \|h\|_{C^3(K)} < \frac{1}{2}\theta \end{array} \right\} \tag{3.53}$$

Naturally, $\widetilde{\mathscr{O}}(\mathsf{H})$ is open and a subset of $\mathscr{O}(\mathsf{H})$. We claim that for the set $\widetilde{\mathscr{O}}(\mathsf{H})$ the assertion of the lemma holds true. Let us fix $\mathsf{h}_1, \mathsf{h}_2 \in \widetilde{\mathscr{O}}(\mathsf{H})$, $\mathsf{h}_1 \neq \mathsf{h}_2$, such that $\mathsf{H} + \mathsf{h}_1, \mathsf{H} + \mathsf{h}_2 \in \mathscr{H}^{reg}$.

Let $\mathcal{F}$ be the coercive function associated to H by property (H4) and denote

$$K' := \{x \in \mathbb{R}^{2n} \mid d_x(d\mathcal{F}(X^{\mathsf{H}}))(X^{\mathsf{H}}(x)) \leq 0\} \cap \{x \in \mathbb{R}^{2n} \mid d_x\mathcal{F}(X^{\mathsf{H}}) = 0\} \cap \mathsf{H}^{-1}(0).$$

This subset of $\mathbb{R}^{2n}$ is compact by Property (H4). Then by Corollary 2.28 for all $h \in C_0^\infty(K)$ we have

$$\forall\ (v, \eta) \in \bigcup_{h \in C_0^\infty(K)} (Crit(\mathcal{A}^{\mathsf{H}+h}) \setminus (\mathcal{A}^{\mathsf{H}+h})^{-1}(0)),$$
$$v(S^1) \subseteq \mathcal{F}^{-1}((-\infty, \sup_{K \cup K'} \mathcal{F}]).$$

In particular, all the Hamiltonians in $H + \mathscr{O}(H)$ satisfy Property (PO+), therefore we can fix a constant $\eta > 0$, such that

$$0 < \eta < \inf\Big\{|\eta| \ \Big| \ (v, \eta) \in \bigcup_{i=1,2} (Crit(\mathcal{A}^{\mathsf{H}+\mathsf{h}_i}) \setminus (\mathcal{A}^{\mathsf{H}+\mathsf{h}_i})^{-1}(0))\Big\}.$$

Moreover, fix an open, precompact subset $\mathcal{V} \subseteq \mathbb{R}^{2n}$, such that

$$\mathcal{F}^{-1}((-\infty, \sup_{K \cup K'} \mathcal{F}]) \subseteq \mathcal{V}.$$

In this way, whenever

$$(v, \eta) \in \bigcup_{i=1,2} (Crit(\mathcal{A}^{\mathsf{H}+\mathsf{h}_i}) \setminus (\mathcal{A}^{\mathsf{H}+\mathsf{h}_i})^{-1}(0)), \text{ then } \qquad v(S^1) \subseteq \mathcal{V} \quad \& \quad |\eta| > \eta.$$

Since $\mathsf{h}_1, \mathsf{h}_2 \in \widetilde{\mathscr{O}}(\mathsf{H}), \mathsf{h}_1 \neq \mathsf{h}_2$ it holds:

$$|\mathsf{h}_2 - \mathsf{h}_1\|_\infty < 2^{-6}\big(\tilde{c} + \frac{1}{\varepsilon_0}\big)^{-1}, \tag{3.54}$$

or equivalently

$$1 < \varepsilon_0(2^{-6}\|\mathsf{h}_2 - \mathsf{h}_1\|_\infty^{-1} - \tilde{c}). \tag{3.55}$$

By Theorem 2 the sets of regular almost complex structures $\mathscr{J}^{reg}(\mathsf{H} + \mathsf{h}_1)$ and $\mathscr{J}^{reg}(\mathsf{H} + \mathsf{h}_2)$ corresponding to $\mathsf{H} + \mathsf{h}_1$ and $\mathsf{H} + \mathsf{h}_2$ respectively are dense in

$$\mathscr{J}\Big(\mathbb{R}^{2n}, \omega_0, \mathcal{V} \times \big((-\infty, \eta) \cup (\eta, +\infty)\big)\Big).$$

Moreover, the set above is simply connected. Therefore, one can find

$$\mathsf{J}_1 \in \mathscr{J}^{reg}(\mathsf{H} + \mathsf{h}_1) \cap \mathscr{J}\Big(\mathbb{R}^{2n}, \omega_0, \mathcal{V} \times \big((-\infty, \eta) \cup (\eta, +\infty)\big)\Big),$$
$$\mathsf{J}_2 \in \mathscr{J}^{reg}(\mathsf{H} + \mathsf{h}_2) \cap \mathscr{J}\Big(\mathbb{R}^{2n}, \omega_0, \mathcal{V} \times \big((-\infty, \eta) \cup (\eta, +\infty)\big)\Big),$$

close enough to the standard almost complex structure J_0, such that there exists a smooth homotopy $\{J_s\}_{s\in\mathbb{R}}$ satisfying

$$\begin{gathered} s \le 0 \quad J_s = \mathsf{J}_1, \qquad s \ge 1 \quad J_s = \mathsf{J}_2, \\ \forall\ s \in \mathbb{R} \quad J_s \in \mathscr{J}\Big(\mathbb{R}^{2n}, \omega_0, \mathcal{V} \times \big((-\infty, \mathfrak{y}) \cup (\mathfrak{y}, +\infty)\big)\Big), \\ \|J\|_\infty^3 < \varepsilon_0 (2^{-6} \|\mathsf{h}_2 - \mathsf{h}_1\|_\infty^{-1} - \tilde{c}). \end{gathered}$$

The last inequality can be obtained due to (3.55).

Let $\beta : \mathbb{R} \to [0,1]$ be a smooth function, such that

$$\begin{aligned} \forall\ s \le 0 \qquad & \beta(s) = 0, \\ \forall\ s \ge 1 \qquad & \beta(s) = 1, \\ \forall\ s \in \mathbb{R} \qquad & \beta'(s) \in [0,2]. \end{aligned}$$

To the two pairs $(\mathsf{h}_1, \mathsf{J}_1)$ and $(\mathsf{h}_2, \mathsf{J}_2)$ we can associate a homotopy $\overline{\Gamma}$ of Hamiltonians and almost complex structures defined as follows:

$$\begin{aligned} \overline{\Gamma} &:= \{(\bar{H}_s, J_s)\}_{s\in\mathbb{R}} \\ \bar{H}_s &:= \mathsf{H} + (1 - \beta(s))\mathsf{h}_1 + \beta(s)\mathsf{h}_2, \qquad \forall\ s \in \mathbb{R} \end{aligned} \tag{3.56}$$

and an open neighborhood of 0 in $C_0^\infty([0,1] \times K)$ defined by:

$$\mathscr{O}(\overline{\Gamma}) := \left\{ \quad h \in C_0^\infty([0,1] \times K) \ \middle|\ \begin{array}{c} \|\partial_s h\|_\infty < 2^{-5}(\tilde{c} + \frac{1}{\varepsilon_0}\{\|J\|_\infty^3, 1\})^{-1} \\ \forall\ s \in [0,1] \quad \|h_s\|_{C^3(K)} < \frac{1}{4}\theta \end{array} \right\} \tag{3.57}$$

For every $h \in \mathscr{O}(\overline{\Gamma})$ we define a homotopy

$$\Gamma(h) := \{(\bar{H}_s + h_s, J_s)\}_{s\in\mathbb{R}}.$$

Combining (3.54), (3.56) and (3.57) we obtain

$$\begin{aligned} \|\partial_s(\bar{H}_s + h_s)\|_\infty &= \|\partial_s(\mathsf{H} + (1 - \beta(s))\mathsf{h}_1 + \beta(s)\mathsf{h}_2 + h_s)\|_\infty \\ &\le \|\beta'\|_\infty \|\mathsf{h}_2 - \mathsf{h}_1\|_\infty + \|\partial_s h\|_\infty \\ &\le 2\|\mathsf{h}_2 - \mathsf{h}_1\|_\infty + \|\partial_s h\|_\infty \\ &< 2^{-4}(\tilde{c} + \frac{1}{\varepsilon_0}\{\|J\|_\infty^3, 1\})^{-1}. \end{aligned} \tag{3.58}$$

Thus we can infer that $\Gamma(h)$ satisfies inequality (3.6). Moreover, by applying (3.53), (3.56) and (3.57) we have

$$\begin{aligned} \max_{s\in[0,1]} \|\bar{H}_s + h_s - \mathsf{H}\| &= \max_{s\in[0,1]} \|(1 - \beta(s))\mathsf{h}_1 + \beta(s)\mathsf{h}_2 + h_s\|_{C^3(K)} \\ &\le \max\{\|\mathsf{h}_1\|_{C^3(K)}, \|\mathsf{h}_2\|_{C^3(K)}\} + \max_{s\in[0,1]} \|h\|_{C^3(K)} \\ &< \frac{3}{4}\theta, \end{aligned} \tag{3.59}$$

which proves that the Hamiltonian part of the homotopy lies in $\mathsf{H} + \mathscr{O}(\mathsf{H})$. In other words, for every $h \in \mathscr{O}(\overline{\Gamma})$ the corresponding homotopy $\Gamma(h)$ satisfies the assumptions of Theorem 7 and as a result one obtains uniform $L^\infty \times \mathbb{R}$ bounds on the corresponding moduli spaces and uniform bounds on the energy as shown in Proposition 3.4. Moreover, using linearity condition between the action and the η as stated in Lemma 3.5 together with inequality (3.6), we can directly apply Corollary 3.8 from [12] to show that every $\Gamma(h)$ satisfies the Novikov conditions. In consequence, the homotopy $\overline{\Gamma}$ and the set $\mathscr{O}(\overline{\Gamma})$ satisfy the assumptions of Lemma 2.15 and we can deduce that for a generic choice of $h \in \mathscr{O}(\overline{\Gamma})$ there exists a homomorphism $\Psi^{\Gamma(h)}$

$$\Psi^{\Gamma(h)} : RFH(\mathsf{H} + \mathsf{h}_2, \mathsf{J}_2) \to RFH(\mathsf{H} + \mathsf{h}_1, \mathsf{J}_1).$$

Note that for every $h \in \mathscr{O}(\overline{\Gamma})$ the inverse homotopy $\Gamma^{-1}(h)$, defined by

$$\Gamma^{-1}(h) := \{(\bar{H}_{1-s} + h_{1-s}, J_{1-s})\}_{s\in\mathbb{R}},$$

also satisfies inequality (3.6) and its Hamiltonian part lies in $\mathsf{H} + \mathscr{O}(\mathsf{H})$. By the same argument as the one above, we can conclude that for a generic choice of $h \in \mathscr{O}(\overline{\Gamma})$ all four homotopies namely $\Gamma(h)$, its inverse $\Gamma^{-1}(h)$, and their concatenations

$$\begin{aligned}
\Gamma(h)\#\Gamma^{-1}(h) &:= \{(\bar{H}_{1-|2s-1|} + h_{1-|2s-1|}, J_{1-|2s-1|})\}_{s\in\mathbb{R}},\\
\Gamma^{-1}(h)\#\Gamma(h) &:= \{(\bar{H}_{|2s-1|} + h_{|2s-1|}, J_{|2s-1|}\}_{s\in\mathbb{R}}
\end{aligned}$$

are regular, i.e. the four homomorphisms in figure below are well defined.

$$\begin{tikzcd}
RFH(\mathsf{H} + \mathsf{h}_1, \mathsf{J}_1) \arrow[loop left, "\Psi^{\Gamma(h)\#\Gamma^{-1}(h)}"] \arrow[rr, bend left, "\Psi^{\Gamma^{-1}(h)}"] & & RFH(\mathsf{H} + \mathsf{h}_2, \mathsf{J}_2) \arrow[loop right, "\Psi^{\Gamma^{-1}(h)\#\Gamma(h)}"] \arrow[ll, bend left, "\Psi^{\Gamma(h)}"]
\end{tikzcd}$$

Now fix such a $\bar{h} \in \mathscr{O}(\overline{\Gamma})$ and construct two corresponding homotopies of homotopies $\widetilde{\Gamma} = \{\widetilde{\Gamma}^\lambda\}_{\lambda\in[0,1]}$ and $\widehat{\Gamma} = \{\widehat{\Gamma}^\lambda\}_{\lambda\in[0,1]}$ by defining for every $\lambda \in [0,1]$

$$\begin{aligned}
\widetilde{\Gamma}^\lambda &:= \{(\bar{H}_{\lambda(1-|2s-1|)} + \bar{h}_{\lambda(1-|2s-1|)}, J_{\lambda(1-|2s-1|)})\}_{s\in\mathbb{R}},\\
\widehat{\Gamma}^\lambda &:= \{(\bar{H}_{\lambda|2s-1|} + \bar{h}_{\lambda|2s-1|}, J_{\lambda|2s-1|})\}_{s\in\mathbb{R}}.
\end{aligned}$$

Then recalling (3.56) we see that $\widetilde{\Gamma}$ and $\widehat{\Gamma}$ satisfy

$$\begin{aligned}
\widetilde{\Gamma}^0 &= \{(\mathsf{H} + \mathsf{h}_1, \mathsf{J}_1)\}_{s\in\mathbb{R}} = Id_{(\mathsf{H}+\mathsf{h}_1,\mathsf{J}_1)}, & \widetilde{\Gamma}^1 &= \Gamma(\mathsf{h})\#\Gamma(\mathsf{h})^{-1},\\
\widehat{\Gamma}^0 &= \{(\mathsf{H} + \mathsf{h}_2, \mathsf{J}_2)\}_{s\in\mathbb{R}} = Id_{(\mathsf{H}+\mathsf{h}_2,\mathsf{J}_2)}, & \widehat{\Gamma}^1 &= \Gamma(\mathsf{h})^{-1}\#\Gamma(\mathsf{h}).
\end{aligned}$$

Moreover, for all $\lambda \in [0,1]$ the corresponding $\widetilde{\Gamma}^\lambda$ and $\widehat{\Gamma}^\lambda$ start and end in $(\mathsf{H}+\mathsf{h}_1,\mathsf{J}_1)$ and $(\mathsf{H}+\mathsf{h}_2,\mathsf{J}_2)$ respectively, i.e. for all $s \in (-\infty,0]\cup[1,+\infty)$

$$(\bar{H}_{\lambda(1-|2s-1|)}+\bar{h}_{\lambda(1-|2s-1|)},J_{\lambda(1-|2s-1|)})=(\mathsf{H}+\mathsf{h}_1,\mathsf{J}_1),$$
$$(\bar{H}_{\lambda|2s-1|}+\bar{h}_{\lambda|2s-1|},J_{\lambda|2s-1|})=(\mathsf{H}+\mathsf{h}_2,\mathsf{J}_2).$$

Now we will construct an open neighborhood of 0 in $C_0^\infty([0,1]^2\times K)$

$$\mathscr{O}(\Gamma(\bar{h})):=\left\{\quad h\in C_0^\infty([0,1]^2\times K)\;\middle|\;\begin{array}{c}\|\partial_s h\|_\infty<\frac{1}{16}(\tilde{c}+\frac{1}{\varepsilon_0}\max\{\|J\|_\infty^3,1\})^{-1}\\ \forall\ (\lambda,s)\in[0,1]^2\quad \|h_s^\lambda\|_{C^3(K)}<\frac{1}{4}\theta\end{array}\right\} \tag{3.60}$$

If we take $h\in\mathscr{O}(\Gamma(\bar{h}))$ and denote

$$\widetilde{\Gamma}^\lambda(h):=\{(\bar{H}_{\lambda(1-|2s-1|)}+\bar{h}_{\lambda(1-|2s-1|)}+h_s^\lambda,J_{\lambda(1-|2s-1|)})\}_{s\in\mathbb{R}},$$
$$\widehat{\Gamma}^\lambda(h):=\{(\bar{H}_{\lambda|2s-1|}+\bar{h}_{\lambda|2s-1|}+h_s^\lambda,J_{\lambda|2s-1|}\}_{s\in\mathbb{R}},$$

then $\{\widetilde{\Gamma}^\lambda(h)\}_{\lambda\in[0,1]}$ is a homotopy of homotopies between $Id_{(\mathsf{H}+\mathsf{h}_1,\mathsf{J}_1)}$ and $\Gamma(\bar{h})\#\Gamma(\bar{h})^{-1}$, whereas $\{\widehat{\Gamma}^\lambda(h)\}_{\lambda\in[0,1]}$ is a homotopy of homotopies between $Id_{(\mathsf{H}+\mathsf{h}_2,\mathsf{J}_2)}$ and $\Gamma^{-1}(\bar{h})\#\Gamma(\bar{h})$.

Now we will show that for every $\lambda\in[0,1]$ and $h\in\mathscr{O}(\Gamma(\bar{h}))$ the corresponding $\widetilde{\Gamma}^\lambda(h)$ satisfies inequality (3.6) and the Hamiltonian part is in $\mathsf{H}+\mathscr{O}(\mathsf{H})$. First observe that for every $\lambda\in[0,1]$ one has

$$\|J_{\lambda(1-|2s-1|)})\|_\infty\le\|J\|_\infty. \tag{3.61}$$

Therefore, for every $\lambda\in[0,1]$ by combining (3.58) and (3.60) we obtain

$$\begin{aligned}\|\partial_s(\bar{H}_{\lambda(1-|2s-1|)}+\bar{h}_{\lambda(1-|2s-1|)}+h_s^\lambda)\|_\infty &\le 2\lambda\|\partial_s(\bar{H}_s+\bar{h}_s)\|_\infty+\|\partial_s h_s)\|_\infty\\ &\le\frac{1}{8}(\tilde{c}+\frac{1}{\varepsilon_0}\max\{\|J\|_\infty^3,1\})^{-1}\\ &\le\frac{1}{8}(\tilde{c}+\frac{1}{\varepsilon_0}\max\{\|J_{\lambda(1-|2s-1|)})|_\infty^3,1\})^{-1},\end{aligned}$$

which proves that for every $\lambda\in[0,1]$ and $h\in\mathscr{O}(\Gamma(\bar{h}))$ the Hamiltonian part of the corresponding $\widetilde{\Gamma}^\lambda(h)$ satisfies inequality (3.6). Moreover, for every $(\lambda,s)\in[0,1]^2$ if we combine (3.59) and (3.60), we get

$$\begin{aligned}&\|\bar{H}_{\lambda(1-|2s-1|)}+\bar{h}_{\lambda(1-|2s-1|)}+h_s^\lambda-\mathsf{H}\|_{C_0^3(K)}\le\\ &\qquad\le\max_{s\in[0,1]}\|\bar{H}_s+\bar{h}_s-\mathsf{H}\|_{C_0^3(K)}+\max_{(\lambda,s)\in[0,1]^2}\|h_s^\lambda\|_{C_0^3(K)}\\ &\qquad<\frac{3}{4}\theta+\frac{1}{4}\theta=\theta,\end{aligned}$$

which assures that for every $\lambda\in[0,1]$ and $h\in\mathscr{O}(\Gamma(\bar{h}))$ the Hamiltonian part of the corresponding homotopy $\widetilde{\Gamma}^\lambda(h)$ is in $\mathsf{H}+\mathscr{O}(\mathsf{H})$.

Using the same arguments as in the proof of Lemma 3.21, we show that for every $h \in \mathscr{O}(\Gamma(\mathsf{h}))$ both families $\{\widetilde{\Gamma}^{\lambda}(h)\}_{\lambda\in[0,1]}$ and $\{\widehat{\Gamma}^{\lambda}(h)\}_{\lambda\in[0,1]}$ satisfy assumptions of Lemma 2.17 and therefore we have:

$$\Psi^{\Gamma(\bar{h})\#\Gamma(\bar{h})^{-1}} = \Psi^{Id_{(\mathsf{H}+\mathsf{h}_1,\mathsf{J}_1)}} = Id,$$
$$\Psi^{\Gamma^{-1}(\bar{h})\#\Gamma(\bar{h})} = \Psi^{Id_{(\mathsf{H}+\mathsf{h}_2,\mathsf{J}_2)}} = Id.$$

To show that the homomorphism corresponding to the concatenation is the composition of homomorphisms, using (3.56) we introduce another pair of families of homotopies, namely:

$$\overline{\Gamma}^R := \{(\bar{H}_{(1+e^{-R})(R+\frac{1}{2}-|s-\frac{1}{2}|)} + \bar{h}_{(1+e^{-R})(R+\frac{1}{2}-|s-\frac{1}{2}|)}, J_{(1+e^{-R})(R+\frac{1}{2}-|s-\frac{1}{2}|)})\}_{s\in\mathbb{R}},$$
$$\widetilde{\Gamma}^R := \{(\bar{H}_{(1+e^{-R})(|s-\frac{1}{2}|-R)} + \bar{h}_{(1+e^{-R})(|s-\frac{1}{2}|-R)}, J_{(1+e^{-R})(|s-\frac{1}{2}|-R)})\}_{s\in\mathbb{R}},$$

where $\bar{H}$ is as in (3.56). Naturally,

$$\overline{\Gamma}^0 = \Gamma(\bar{h})\#\Gamma^{-1}(\bar{h}), \qquad \text{and} \qquad \widetilde{\Gamma}^0 = \Gamma^{-1}(\bar{h})\#\Gamma(\bar{h}).$$

Unfortunately, for $R > 0$ the homotopies $\overline{\Gamma}^R$ and $\widetilde{\Gamma}^R$ are not in the same class of homotopies as in Theorem 7, therefore we cannot apply it directly. However, they are constant outside of a compact set and the measure of the set where $\overline{\Gamma}^R$ and $\widetilde{\Gamma}^R$ are non-constant is uniformly bounded. Indeed, one can calculate that for every $R \geq 0$

$$\begin{aligned}
supp(\partial_s J_{(1+e^{-R})(R+\frac{1}{2}-|s-\frac{1}{2}|)}) &= supp(\partial_s(\bar{H}_{(1+e^{-R})(R+\frac{1}{2}-|s-\frac{1}{2}|)} + \bar{h}_{(1+e^{-R})(R+\frac{1}{2}-|s-\frac{1}{2}|)})) \\
&\subseteq [-R, -R+1] \cup [R, R+1] \\
supp(\partial_s(J_{(1+e^{-R})(|s-\frac{1}{2}|-R)})) &= supp(\partial_s(\bar{H}_{(1+e^{-R})(|s-\frac{1}{2}|-R)} + \bar{h}_{(1+e^{-R})(|s-\frac{1}{2}|-R)})) \\
&\subseteq [-R - \frac{1}{2}] \cup [R + \frac{1}{2}, R + \frac{3}{2}].
\end{aligned}$$

That means that for every $R \geq 0$ the homotopies $\overline{\Gamma}^R$ and $\widetilde{\Gamma}^R$ are constant outside of a compact set and the measure of the sets where $\overline{\Gamma}^R$ and $\widetilde{\Gamma}^R$ are non-constant is bounded by 2. Therefore, we claim, as in the proof of Lemma 3.21, that the families $\{\overline{\Gamma}^R\}_{R\geq 0}$ and $\{\widetilde{\Gamma}^R\}_{R\geq 0}$ exhibit uniform bounds on the energy and moduli spaces and therefore we can apply Proposition 2.19, which gives us:

$$\Psi^{\Gamma(\bar{h})\#\Gamma^{-1}(\bar{h})} = \Psi^{\Gamma(\bar{h})} \circ \Psi^{\Gamma^{-1}(\bar{h})},$$
$$\Psi^{\Gamma^{-1}(\bar{h})\#\Gamma(\bar{h})} = \Psi^{\Gamma^{-1}(\bar{h})} \circ \Psi^{\Gamma(\bar{h})}.$$

and concludes the proof that

$$\Psi^{\Gamma(\bar{h})} : RFH(\mathsf{H}+\mathsf{h}_2, \mathsf{J}_2) \to RFH(\mathsf{H}+\mathsf{h}_1, \mathsf{J}_1)$$

is an isomorphism. □